

# ALMOST SURE BEHAVIOR FOR THE LOCAL TIME OF A DIFFUSION IN A SPECTRALLY NEGATIVE LÉVY ENVIRONMENT

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**ABSTRACT.** We study the almost sure asymptotic behavior of the supremum of the local time for a transient diffusion in a spectrally negative Lévy environment. In particular, we link this behavior with the left tail of an exponential functional of the environment conditioned to stay positive.

## 1. INTRODUCTION

We study the almost sure asymptotic behavior of the supremum of the local time for a transient diffusion in a spectrally negative Lévy environment. Let  $(V(x), x \in \mathbb{R})$  be a spectrally negative Lévy process on  $\mathbb{R}$  which is not the opposite of a subordinator, drifts to  $-\infty$  at  $+\infty$ , and such that  $V(0) = 0$ . We denote its Laplace exponent by  $\Psi_V$  :

$$\forall t, \lambda \geq 0, \mathbb{E} \left[ e^{\lambda V(t)} \right] = e^{t\Psi_V(\lambda)}.$$

It is well-known, for such  $V$ , that  $\Psi_V$  admits a non trivial zero that we denote here by  $\kappa$ ,  $\kappa := \inf\{\lambda > 0, \Psi_V(\lambda) = 0\} > 0$ .

We are here interested in a diffusion in this potential  $V$ . Such a diffusion  $(X(t), t \geq 0)$  is defined informally by  $X(0) = 0$  and

$$dX(t) = d\beta(t) - \frac{1}{2}V'(X(t))dt,$$

where  $\beta$  is a Brownian motion independent from  $V$ . Rigorously,  $X$  is defined by its conditional generator given  $V$ ,

$$\frac{1}{2}e^{V(x)} \frac{d}{dx} \left( e^{-V(x)} \frac{d}{dx} \right).$$

The fact that  $V$  drifts to  $-\infty$  puts us in the case where the diffusion  $X$  is a.s. transient to the right. The asymptotic behavior of this diffusion has been studied by Singh [12], he distinguishes three main possible behaviors depending on  $0 < \kappa < 1$ ,  $\kappa = 1$  or  $\kappa > 1$  (the case  $\kappa > 1$  being also divided into three subcases). We denote by  $(\mathcal{L}_X(t, x), t \geq 0, x \in \mathbb{R})$  the version of the local time that is continuous in time and càd-làg in space, and we define the supremum of the local time until instant  $t$  as

$$\mathcal{L}_X^*(t) = \sup_{x \in \mathbb{R}} \mathcal{L}_X(t, x).$$

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*Date:* 24/03/2017 à 00:28:51.

*2010 Mathematics Subject Classification.* 60K37, 60J55, 60G51.

*Key words and phrases.* Diffusion, random potential, Lévy process, renewal process, local time, Lévy processes conditioned to stay positive, exponential functionals.

Here, we study the almost sure asymptotic behavior of  $\mathcal{L}_X^*(t)$ . When the environment is a brownian motion with no drift, Shi [11] has studied this behavior and shown that

$$\mathbb{P}\text{-a.s.} \limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t \log(\log(\log(t)))} \geq \frac{1}{32}, \quad (1.1)$$

where  $\mathbb{P}$  is the so-called annealed probability measure which definition is recalled in Subsection 1.3. In the same case, Androletti and Diel [3] have proved more recently the convergence in distribution of  $\mathcal{L}_X^*(t)/t$ . Diel [7] has then continued the study by giving a finite upper bound for the lim sup in (1.1) and doing the same study for the lim inf :

$$\mathbb{P}\text{-a.s.} \limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t \log(\log(\log(t)))} \leq \frac{e^2}{2} \quad \text{and} \quad \frac{j_0^2}{64} \leq \liminf_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t / \log(\log(\log(t)))} \leq \frac{e^2 \pi^2}{4},$$

where  $j_0$  is the smallest positive root of the Bessel function  $J_0$ .

In the case of a drifted-brownian environment, the almost sure behavior of  $\mathcal{L}_X^*(t)$  has been studied by Devulder [6] using annealed methods. He totally characterizes the almost sure behavior when  $\kappa > 1$ . In this case, for any positive non decreasing function  $a$  we have

$$\sum_{n=1}^{+\infty} \frac{1}{n(a(n))} \begin{cases} < +\infty \\ = +\infty \end{cases} \Leftrightarrow \limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{(ta(t))^{1/\kappa}} = \begin{cases} 0 \\ +\infty \end{cases} \mathbb{P}\text{-a.s.}, \quad (1.2)$$

and

$$\mathbb{P}\text{-a.s.} \liminf_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{(t / \log(\log(t)))^{1/\kappa}} = 4(\kappa^2(\kappa - 1)/8)^{1/\kappa}. \quad (1.3)$$

When  $\kappa = 1$  he obtains

$$\mathbb{P}\text{-a.s.} \liminf_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t / \log(t) \log(\log(t))} \leq 1/2.$$

When  $0 < \kappa < 1$ , his method fails and only provides partial results for the almost sure behavior of the local time. More precisely he proves that the renormalisation for the lim sup is greater than  $t$  :

$$\mathbb{P}\text{-a.s.} \limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t} = +\infty, \quad (1.4)$$

and that the renormalization for the lim inf is at most  $t / \log(\log(t))$  and greater than  $t / (\log(t))^{1/\kappa} (\log(\log(t)))^{(2/\kappa)+\epsilon}$  for any  $\epsilon > 0$  :

$$\mathbb{P}\text{-a.s.} \liminf_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t / \log(\log(t))} \leq C(\kappa), \quad (1.5)$$

$$\forall \epsilon > 0, \mathbb{P}\text{-a.s.} \liminf_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t / (\log(t))^{1/\kappa} (\log(\log(t)))^{(2/\kappa)+\epsilon}} = +\infty, \quad (1.6)$$

where  $C(\kappa)$  is a positive non explicit constant.

For the discrete transient *Random Walk in Random Environment* (RWRE), the almost sure behavior of the supremum of the local time has been studied by Gantert and Shi [10], they obtain the behavior of the lim sup in the two subcases  $0 < \kappa \leq 1$  and  $\kappa > 1$ .

For the diffusion in the general potential  $V$ , the convergence in distribution of  $\mathcal{L}_X^*(t)/t$  has been studied by the author in [14] using two different methods : when  $0 < \kappa < 1$  a path decomposition of the environment that provides an interesting renewal structure to study the diffusion is used, this method is inspired from Androletti and al. [2], [1] which are themselves inspired from the work of Enriquez and al. [9] in the discrete case. When  $\kappa > 1$  an equality in law between the

local time and a generalized Ornstein-Uhlenbeck process that was introduced in [12] is used. The main contribution of this paper is in the case  $0 < \kappa < 1$ , pushing further the ideas of [14], we make a deep study of the renewal structure of the diffusion in order to establish the almost sure asymptotic behavior of  $\mathcal{L}_X^*(t)$ . We characterize this behavior when  $0 < \kappa < 1$  and  $\kappa > 1$ . In particular, the restriction of our results to the case of a drifted brownian potential with  $0 < \kappa < 1$  improves the results (1.4), (1.5) and (1.6) of [6] by giving the exact renormalizations and even the exact value of the constant for the lim sup, for the lim inf we get the exact renormalizations and an explicit upper bound of the constant.

**1.1. Main results.** We start with the case  $0 < \kappa < 1$ . In that case, the limit distribution of  $\mathcal{L}_X^*(t)/t$  given in [14] depends on exponential functionals of  $V$  and its dual conditioned to stay positive. They are defined as follow :

$$I(V^\uparrow) := \int_0^{+\infty} e^{-V^\uparrow(t)} dt \quad \text{and} \quad I(\hat{V}^\uparrow) := \int_0^{+\infty} e^{-\hat{V}^\uparrow(t)} dt,$$

where  $\hat{V}$ , the dual of  $V$ , is equal in law to  $-V$ , and where  $V^\uparrow$  and  $\hat{V}^\uparrow$  denote respectively  $V$  and  $\hat{V}$  conditioned to stay positive. In Subsection 4.1 it is precised how  $V^\uparrow$  and  $\hat{V}^\uparrow$  are defined rigorously. These functionals are studied by the author in [13] where it is proved in Theorems 1.1 and 1.13 that they are indeed finite and well-defined. Let  $G_1$  and  $G_2$  be two independent random variables with  $G_1 \stackrel{\mathfrak{L}}{=} I(V^\uparrow)$  and  $G_2 \stackrel{\mathfrak{L}}{=} I(\hat{V}^\uparrow)$ . We define  $\mathcal{R} := G_1 + G_2$ .

To study the lim sup of  $\mathcal{L}_X^*(t)$ , we link the almost sur asymptotic behavior of  $\mathcal{L}_X^*(t)$  with the left tail of  $I(V^\uparrow)$  (or of  $\mathcal{R}$  according to the case). Before stating our results, let us recall what is known about the left tail of  $I(V^\uparrow)$ .

In [13], the left tail of  $I(V^\uparrow)$  is linked to the asymptotic behavior of  $\Psi_V$ . This asymptotic behavior is usually quantified thanks to two real numbers  $\sigma$  and  $\beta$  :

$$\begin{aligned} \sigma &:= \sup \left\{ \alpha \geq 0, \lim_{\lambda \rightarrow +\infty} \lambda^{-\alpha} \Psi_V(\lambda) = \infty \right\}, \\ \beta &:= \inf \left\{ \alpha \geq 0, \lim_{\lambda \rightarrow +\infty} \lambda^{-\alpha} \Psi_V(\lambda) = 0 \right\}. \end{aligned}$$

If  $\Psi_V$  has  $\alpha$ -regular variation for  $\alpha \in [1, 2]$  (for example if  $V$  is a drifted  $\alpha$ -stable Lévy process with no positive jumps), we have  $\sigma = \beta = \alpha$ . Note that when  $Q$ , the brownian component of  $V$ , is positive, then  $\Psi_V$  has 2-regular variation, and when  $Q = 0$ ,  $1 \leq \sigma \leq \beta \leq 2$ . The asymptotic behavior of  $\mathbb{P}(I(V^\uparrow) \leq x)$  as  $x$  goes to 0 is given by the following theorem from [13] :

**Theorem 1.1.** [Véchambre, [13]]

*There is a positive constant  $K_0$  (depending on  $V$ ) such that for  $x$  small enough*

$$\mathbb{P} \left( I(V^\uparrow) \leq x \right) \leq e^{-K_0/x}. \quad (1.7)$$

*More precisely we have*

$$\forall l < \frac{1}{\beta - 1}, \quad \mathbb{P} \left( I(V^\uparrow) \leq x \right) \leq e^{-1/x^l}. \quad (1.8)$$

$$\text{If } \sigma > 1, \quad \forall l > \frac{1}{\sigma - 1}, \quad \mathbb{P} \left( I(V^\uparrow) \leq x \right) \geq e^{-1/x^l}. \quad (1.9)$$

If there are two positive constants  $c < C$  and  $\alpha \in ]1, 2]$  such that  $c\lambda^\alpha \leq \Psi_V(\lambda) \leq C\lambda^\alpha$  for  $\lambda$  large enough, then for any  $\delta > 1$  we have, when  $x$  is small enough,

$$\exp\left(-\frac{\delta\alpha^{\frac{\alpha}{\alpha-1}}}{(cx)^{\frac{1}{\alpha-1}}}\right) \leq \mathbb{P}\left(I(V^\uparrow) \leq x\right) \leq \exp\left(-\frac{\alpha-1}{\delta(Cx)^{\frac{1}{\alpha-1}}}\right). \quad (1.10)$$

If there is a positive constant  $C$  and  $\alpha \in ]1, 2]$  such that  $\Psi_V(\lambda) \sim_{\lambda \rightarrow +\infty} C\lambda^\alpha$ , then

$$-\log\left(\mathbb{P}\left(I(V^\uparrow) \leq x\right)\right) \underset{x \rightarrow 0}{\sim} \frac{\alpha-1}{(Cx)^{\frac{1}{\alpha-1}}}. \quad (1.11)$$

(1.7) is Remark 1.7 of [13], (1.8) and (1.9) are a reformulation of Theorem 1.4 of [13], (1.10) comes from Theorem 1.2 of [13], and (1.11) is Corollary 1.6 of [13].

We can now state our results for the lim sup :

**Theorem 1.2.** • Assume that  $0 < \kappa < 1$ ,  $V$  has unbounded variation,  $V(1) \in L^p$  for some  $p > 1$  and  $V$  possesses negative jumps.

If there is  $\gamma > 1$  and  $C > 0$  such that for  $x$  small enough

$$\mathbb{P}\left(I(V^\uparrow) \leq x\right) \leq \exp\left(-\frac{C}{x^{\frac{1}{\gamma-1}}}\right), \quad (1.12)$$

then we have

$$\mathbb{P}\text{-a.s.} \limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t(\log(\log(t)))^{\gamma-1}} \leq C^{1-\gamma}. \quad (1.13)$$

If there is  $\gamma > 1$  and  $C > 0$  such that for  $x$  small enough

$$\mathbb{P}\left(I(V^\uparrow) \leq x\right) \geq \exp\left(-\frac{C}{x^{\frac{1}{\gamma-1}}}\right), \quad (1.14)$$

then we have

$$\mathbb{P}\text{-a.s.} \limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t(\log(\log(t)))^{\gamma-1}} \geq C^{1-\gamma}. \quad (1.15)$$

- Assume now that  $V(t) = W_\kappa(t) := W(t) - \frac{\kappa}{2}t$  with  $0 < \kappa < 1$ , (i.e.  $V$  is the  $\kappa$ -drifted brownian motion), then the above implications ((1.12)  $\Rightarrow$  (1.13) and (1.14)  $\Rightarrow$  (1.15)) are still true with  $I(V^\uparrow)$  replaced by  $\mathcal{R}$ .

In the above theorem we had to distinguish the case where  $V$  possesses negative jumps and the case where  $V$  is a drifted brownian motion. First, note that this is a true alternative : since  $V$  is spectrally negative, the case where  $V$  do not possess negative jumps is the case where  $V$  do not possess jumps at all and is therefore a drifted brownian motion. In this case we assumed for convenience that the gaussian component of  $V$  is normalized to 1. The difference between the two cases in the above proposition comes from the absence or presence of symmetry for the environment. When  $V$  possesses negative jumps,  $\hat{V}^\uparrow$  possesses positives jumps that might repulse it very fast from 0, this is why the left tail of  $I(V^\uparrow)$  is thinner than the left tail of  $I(\hat{V}^\uparrow)$ , as we can see from the results of [13] (comparing Remark 1.7 of [13] with the combination of Proposition 1.15 and Theorem 1.16 of [13]). As a consequence, only the left tail of  $I(V^\uparrow)$  is relevant in the left tail of  $\mathcal{R}$ . When  $V$  is a drifted brownian motion, a symmetry appears :  $\hat{V}^\uparrow$  and  $V^\uparrow$  are equal in law,  $\mathcal{R}$  is then the sum of two independent random variables having the same law as  $I(V^\uparrow)$  and none of them can be neglected.

**Remark 1.3.** *It has to be noted that the  $\limsup$  above is  $\mathbb{P}$ -almost surely equal to a constant belonging to  $[0, +\infty]$  and that the inequalities (1.13) and (1.15) are inequalities relative to this value that the  $\limsup$  equals  $\mathbb{P}$ -almost surely. The same will be true in all the results below : all the  $\limsup$  and  $\liminf$  considered are  $\mathbb{P}$ -almost surely equal to a constant. This fact is justified in Subsection 4.4.*

Putting together Theorem 1.2 and what is known for the left tail of  $I(V^\uparrow)$  (Theorem 1.1), we can state precise results for the  $\limsup$  :

**Theorem 1.4.** *If  $0 < \kappa < 1$ ,  $V$  has unbounded variation and  $V(1) \in L^p$  for some  $p > 1$ , then we have*

$$\forall \beta' > \beta, \mathbb{P}\text{-a.s.} \limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t(\log(\log(t)))^{\beta'-1}} = 0, \quad (1.16)$$

and

$$\text{If } \sigma > 1, \forall \sigma' \in ]1, \sigma[, \mathbb{P}\text{-a.s.} \limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t(\log(\log(t)))^{\sigma'-1}} = +\infty. \quad (1.17)$$

If we make further hypothesis on the regularity of the variation of  $\Psi_V$  we can give the exact order of  $\mathcal{L}_X^*(t)$  :

**Theorem 1.5.** • *Assume that  $0 < \kappa < 1$ ,  $V$  has unbounded variation,  $V(1) \in L^p$  for some  $p > 1$  and  $V$  possesses negative jumps.*

*If there are two positive constants  $c < C$  and  $\alpha \in ]1, 2]$  such that  $c\lambda^\alpha \leq \Psi_V(\lambda) \leq C\lambda^\alpha$  for  $\lambda$  large enough, then we have*

$$\mathbb{P}\text{-a.s.} \frac{c}{\alpha^\alpha} \leq \limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t(\log(\log(t)))^{\alpha-1}} \leq \frac{C}{(\alpha-1)^{\alpha-1}}.$$

*If, more precisely, there is a positive constant  $C$  and  $\alpha \in ]1, 2]$  such that  $\Psi_V(\lambda) \sim C\lambda^\alpha$  for large  $\lambda$ , then we have*

$$\mathbb{P}\text{-a.s.} \limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t(\log(\log(t)))^{\alpha-1}} = \frac{C}{(\alpha-1)^{\alpha-1}}.$$

• *Assume now that  $V = W_\kappa$ , the  $\kappa$ -drifted brownian motion, with  $0 < \kappa < 1$ , then we have*

$$\mathbb{P}\text{-a.s.} \limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t(\log(\log(t)))} = \frac{1}{8}.$$

**Remark 1.6.** *According to the combination of Theorem 1.2 and (1.7) we see that, if  $0 < \kappa < 1$ ,  $V$  has unbounded variation and  $V(1) \in L^p$  for some  $p > 1$ , then we always have*

$$\mathbb{P}\text{-a.s.} \limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t(\log(\log(t)))} < +\infty.$$

*In other words,  $t(\log(\log(t)))$  is the maximal possible renormalisation for the  $\limsup$ .*

**Remark 1.7.** *As it was noticed by Shi [11] and Diel [7] for the recurrent case, we also notice a difference between the renormalization of the local time for the discrete transient RWRE with zero speed (given by Gantert and Shi [10]) and the renormalization of the local time that we give for the transient diffusion with zero speed. This difference can be explained as in the recurrent case : the valleys can potentially be much steeper in the continuous case with a potential having unbounded variation than in the discrete case, so the local maxima of the local time can potentially be higher in the first case.*

We see in the above two theorems that the renormalization of  $\mathcal{L}_X^*(t)$  for the limsup depends directly on the asymptotic behavior of  $\Psi_V$ . In particular, Theorem 1.5 says that for a drifted  $\alpha$ -stable environment (with no positive jumps and  $\alpha > 1$ ), the renormalization of  $\mathcal{L}_X^*(t)$  is  $t(\log(\log(t)))^{\alpha-1}$ . We see that we have much more possible behaviors with general spectrally negative Lévy environments, than with drifted brownian environments. Even if, for technical reasons, the above theorems do not apply when the environment  $V$  has bounded variation, we can conjecture that the behavior of  $\mathcal{L}_X^*(t)$  remains linked in the same way to the left tail of  $I(V^\uparrow)$  which is given by Remark 1.8 of [13]. This implies

**Conjecture 1.8.** *When  $V$  has bounded variation, we have*

$$\mathbb{P}\text{-a.s. } 0 < \limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t} < +\infty.$$

If this conjecture is true, we would have, when the environment has bounded variation, the same renormalization as in the discrete transient case given by Theorem 1.1 of [10]. This would not be surprising since the discrete case gives rise to potentials of bounded variation. Moreover, if  $V$  has bounded variation then it is known to be the difference of a deterministic positive drift and a subordinator. The valleys can then not be steeper than the deterministic drift so, according to Remark 1.7, the expected renormalization of  $\mathcal{L}_X^*(t)$  has to be the same as in the discrete case.

For the liminf, there is only one possible renormalization. Our result is as follows :

**Theorem 1.9.** *If  $0 < \kappa < 1$ ,  $V$  has unbounded variation and  $V(1) \in L^p$  for some  $p > 1$ , then we have*

$$\mathbb{P}\text{-a.s. } 0 < \liminf_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t / \log(\log(t))} \leq \frac{1 - \kappa}{\kappa(\mathbb{E}[I(V^\uparrow)] + \mathbb{E}[I(\hat{V}^\uparrow)])}. \quad (1.18)$$

Note that the expectations  $\mathbb{E}[I(V^\uparrow)]$  and  $\mathbb{E}[I(\hat{V}^\uparrow)]$  are finite and well defined since  $I(V^\uparrow)$  and  $I(\hat{V}^\uparrow)$  both admit some finite exponential moments according to Theorems 1.1 and 1.13 of [13].

**Example :** We consider  $W_\kappa$  the  $\kappa$ -drifted brownian motion ( $W_\kappa(t) := W(t) - \frac{\kappa}{2}t$ ), then, the expression of the Laplace transform of  $I(W_\kappa^\uparrow)$  is given by equation (1.12) of [13]. This expression allows to compute the moments of  $I(W_\kappa^\uparrow)$  and gives in particular  $\mathbb{E}[I(W_\kappa^\uparrow)] = 2/(1+\kappa)$ . Moreover  $W_\kappa^\uparrow$  and  $\hat{W}_\kappa^\uparrow$  have the same law so  $\mathbb{E}[I(W_\kappa^\uparrow)] + \mathbb{E}[I(\hat{W}_\kappa^\uparrow)] = 4/(1+\kappa)$ . If we choose, as an environment,  $V = W_\kappa$  (for  $0 < \kappa < 1$ ), then the above upper bound for the liminf becomes  $(1-\kappa^2)/4\kappa$ . Putting this in relation with the results of [6], we see that the application of Theorem 1.9 in the special case of a drifted brownian environment improves (1.6) and completes (1.5) by proving that this renormalization is exact and by providing an explicit upper bound.

The fact that we have many possible renormalizations for the limsup, depending on the environment  $V$ , while only one for the liminf, whatever is the environment  $V$ , might seem surprising, here is an heuristic explanation : In each valley the contribution to the time equals approximately the contribution to the local time multiplied by an exponential functional of the bottom of the valley (which is close to  $\mathcal{R}$ ). The limsup concerns large values of the local time at a fixed time, it is reached when the contribution to the local time of some valley is large while the contribution to the time of the same valley has a fixed value, this happens when the exponential functional of the bottom of the valley has a small value. The link between the limsup and the small values of an exponential functional is made rigorously in Theorem 1.2. The liminf concerns small values of the local time at a fixed time, it is reached when the contributions to the local time of the valleys are small while the sum of their contributions to the time have a fixed value, this happens when the exponential functionals of the bottoms of some valleys are large. We see that the difference between limsup and liminf comes from the difference between the left and

right tails of  $\mathcal{R}$ . The left tail is mainly the left tail of  $I(V^\uparrow)$  which depends on the asymptotic of  $\Psi_V$ , according to Theorem 1.1, and  $\Psi_V$  have many possible behaviors. On the other hand, the right tail is always exponential according to Theorems 1.1 and 1.13 of [13]. This explains the difference of behaviors between the  $\limsup$  and the  $\liminf$ .

We now treat the case where  $\kappa > 1$ . In this case we use a different method and the results are different.

**Theorem 1.10.** *Let  $f$  be a positive non-increasing function. When  $\kappa > 1$ , we have*

$$\int_1^{+\infty} \frac{(f(t))^\kappa}{t} dt \begin{cases} < +\infty \\ = +\infty \end{cases} \Leftrightarrow \limsup_{t \rightarrow +\infty} \frac{f(t) \mathcal{L}_X^*(t)}{t^{1/\kappa}} = \begin{cases} 0 \\ +\infty \end{cases} \mathbb{P}\text{-a.s.}$$

The above result is the analogue of Theorem 1.2 of [10] for the continuous case. In the special case where  $V = W_\kappa$  (for  $\kappa > 1$ ), our result coincides with (1.2) (proved by Devulder in [6]). Indeed, since  $f$  is decreasing we have the easy equivalences

$$\int_1^{+\infty} \frac{(f(t))^\kappa}{t} dt < +\infty \Leftrightarrow \sum_{n=1}^{+\infty} \frac{(f(n))^\kappa}{n} < +\infty \Leftrightarrow \sum_{n=1}^{+\infty} (f(2^n))^\kappa < +\infty. \quad (1.19)$$

The first equivalence shows that our result agrees with (1.2) and the second equivalence allows to reformulate the integrability condition in a form that is convenient to prove Theorem 1.10.

Comparing Theorems 1.4, 1.5 and 1.10 we see that the renormalization for the  $\limsup$  is larger in the slow transient case than in the fast transient case : this is in accordance with intuition. However, it is surprising to see that the renormalization in the slow transient case is also greater than the renormalization in the recurrent case, given by Theorem 1.1 of [7]. Here is the heuristic explanation : In the recurrent case the diffusion is trapped in the bottom of a large valley while in the slow transient case the diffusion gets successively trapped in the bottom of many valleys, these bottom being much more narrow. This explains that the large values of the local time have the tendency to be higher in the second case.

For the  $\liminf$ , we provide an explicite value. Let the constants  $K$  and  $m$  be defined similarly as in [12] :

$$K := \mathbb{E} \left[ \left( \int_0^{+\infty} e^{V(t)} dt \right)^{\kappa-1} \right] \quad \text{and} \quad m := \frac{-2}{\Psi_V(1)} > 0.$$

We have :

**Theorem 1.11.** *When  $\kappa > 1$ , we have*

$$\mathbb{P}\text{-a.s.} \quad \liminf_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{(t/\log(\log(t)))^{1/\kappa}} = 2(\Gamma(\kappa)\kappa^2 K/m)^{1/\kappa}.$$

**Example :** If we choose  $V = W_\kappa$  (for  $\kappa > 1$ ), then  $K = 2^{\kappa-1}/\Gamma(\kappa)$  (see Example 1.1 in [12]) and  $m = 4/(\kappa - 1)$ . The above limit is then  $4(\kappa^2(\kappa - 1)/8)^{1/\kappa}$ . This coincides with (1.3) (proved by Devulder in [6]).

For the  $\liminf$  the behaviors of the different cases are in accordance with intuition : comparing Theorem 1.1 of [7] with Theorems 1.9 and 1.11 we see that the renormalization for the  $\liminf$  in the fast transient case is smaller than the renormalization in the slow transient case which is in turn smaller than the renormalization in the recurrent case.



**1.2. Sketch of proofs and organisation of the paper.** The rest of the paper is organized as follows.

In section 2 we study the case  $0 < \kappa < 1$ . We first recall the decomposition of the environment into valleys from [14] and the behavior of the diffusion with respect to these valleys. In particular, we recall how the renewal structure of the diffusion allows to approximate the supremum of the local time and the time spent by the diffusion in the bottom of the valleys by an *iid* sequence of  $\mathbb{R}^2$ -valued random variables.

For the limsup, we study the asymptotic of the distribution function  $\mathbb{P}(\mathcal{L}_X^*(t)/t \geq x_t)$ , where  $x_t$  is a suitably chosen quantity that goes to infinity with  $t$ . More precisely, in Proposition 2.11 we compare the asymptotic of this distribution function with the one of  $\mathbb{P}(Y_1^{\natural,t}(Y_2^{-1,t}(1)-) \geq x_t)$ , the distribution function of a functional of the above mentioned *iid* sequence, and in Proposition 2.15 we link the asymptotic behavior of  $\mathbb{P}(Y_1^{\natural,t}(Y_2^{-1,t}(1)-) \geq x_t)$  with the left tail of  $I(V^\uparrow)$  (or  $\mathcal{R}$  in the case of a drifted brownian potential). The synthesis of Propositions 2.11 and 2.15 allows to compare the distribution function  $\mathbb{P}(\mathcal{L}_X^*(t)/t \geq x_t)$  with the left tail of  $I(V^\uparrow)$  (or  $\mathcal{R}$  in the case of a drifted brownian potential) in Proposition 2.16. This proposition entails Theorem 1.2 by the mean of the Borel-Cantelli Lemma and the technical Lemma 2.10, which decomposes the trajectory of the diffusion into large independent parts in order to get the required independence to apply the Borel-Cantelli Lemma. The combination of Theorem 1.2 with what is known (and recalled in Theorem 1.1) for the left tail of  $I(V^\uparrow)$  easily yields Theorems 1.4 and 1.5 which solves the problem for the limsup.

For the liminf, we study the quantity  $\mathbb{P}(\mathcal{L}_X^*(t) \leq t/x_t)$ . In Proposition 2.17 it is compared with  $\mathbb{P}(Y_1^{\natural,t}(Y_2^{-1,t}(1)) \leq 1/x_t)$ , the distribution function of an other functional of the  $\mathbb{R}^2$ -valued *iid* sequence. In Lemmas 2.18 and 2.19 we study the Laplace transform of a random variable involved in this functional, this allows to give a lower and an upper bound for  $\mathbb{P}(Y_1^{\natural,t}(Y_2^{-1,t}(1)) \leq 1/x_t)$  in Proposition 2.20. The synthesis of Propositions 2.17 and 2.20 gives the asymptotic of  $\mathbb{P}(\mathcal{L}_X^*(t) \leq t/x_t)$  in Proposition 2.21. This Proposition entails Theorem 1.9 by the mean of the Borel-Cantelli Lemma and Lemma 2.10. This solves the problem for the liminf.

In Section 3 we study the case  $\kappa > 1$ . In this case, the local time at  $t$  can be approximated by the local time at an hitting time and the latter has the same law as the generalized Ornstein-Uhlenbeck process introduced in [12]. Using what is known for the excursion measure of this process we prove Theorems 1.10 and 1.11.

In Section 4 we justify some facts about  $V$ ,  $V^\uparrow$  and the diffusion in  $V$  that are used along the paper.

**1.3. Facts and notations.** For  $Y$  a process and  $S$  a borelian set, we denote

$$\tau(Y, S) := \inf \{t \geq 0, Y(t) \in S\}, \quad \mathcal{K}(Y, S) := \sup \{t \geq 0, Y(t) \in S\}.$$

We shall only write  $\tau(Y, x)$  instead of  $\tau(Y, \{x\})$  and  $\tau(Y, x+)$  instead of  $\tau(Y, [x, +\infty[)$ . Since  $V$  has no positive jumps we see that each positive level is reached continuously (or not reached at all) :  $\forall x > 0, \tau(V, x+) = \tau(V, x)$  (which is possibly infinite). Moreover, the law of the supremum of  $V$  is known, it is an exponential distribution with parameter  $\kappa$  (see Corollary VII.2 in Bertoin [4]).

If  $Y$  is Markovian and  $x \in \mathbb{R}$  we denote  $Y_x$  for the process  $Y$  starting from  $x$ . For  $Y_0$  we shall only write  $Y$ . When it exists we denote by  $(\mathcal{L}_Y(t, x), t \geq 0, x \in \mathbb{R})$  the version of the local time that is continuous in time and càd-làg in space and by  $(\sigma_Y(t, x), t \geq 0, x \in \mathbb{R})$  the inverse of the local time :  $\sigma_Y(t, x) := \inf \{s \geq 0, \mathcal{L}_Y(s, x) > t\}$ .



Let  $B$  be a brownian motion starting at 0 and independent from  $V$ . A diffusion in potential  $V$  can be defined via the formula :

$$X(t) := A_V^{-1}(B(T_V^{-1}(t))). \quad (1.20)$$

where

$$A_V(x) := \int_0^x e^{V(u)} du \text{ and for } 0 \leq s \leq \tau \left( B, \int_0^{+\infty} e^{V(u)} du \right), \quad T_V(s) := \int_0^s e^{-2V(A_V^{-1}(B(u)))} du.$$

It is known that the local time of  $X$  at  $x$  until instant  $t$  has the following expression :

$$\mathcal{L}_X(t, x) = e^{-V(x)} \mathcal{L}_B(T_V^{-1}(t), A_V(x)). \quad (1.21)$$

Recall the notation  $\mathcal{L}_X^*(t)$  for the supremum of the local time until time  $t$ . We also sometimes use the notation  $\mathcal{L}_X^{*,+}$  for the supremum of the local time on the positive half-line :  $\mathcal{L}_X^{*,+}(t) := \sup_{x \in [0, +\infty[} \mathcal{L}_X(t, x)$ .

For the hitting times of  $r \in \mathbb{R}$  by the diffusion  $X$  we shall use the frequent notation  $H(r)$  (instead of  $\tau(X, r)$ ).

$D(\mathbb{R}, \mathbb{R})$  is the space of càd-làg functions from  $\mathbb{R}$  to  $\mathbb{R}$ . Let  $P$  be the probability measure on  $D(\mathbb{R}, \mathbb{R})$  inducing the law of  $V$ . For  $v \in D(\mathbb{R}, \mathbb{R})$ , the quenched probability measure  $P^v$  is the probability measure (associated with the diffusion  $X$ ) conditionally on  $\{V = v\}$ .  $\mathbb{P}$  represents the annealed probability measure, it is defined as  $\mathbb{P}(\cdot) := \int_{D(\mathbb{R}, \mathbb{R})} P^v(\cdot) P(dv)$ .  $X$  is a Markovian process under  $P^v$  but not under  $\mathbb{P}$ . Note that all the almost sure convergences stated in this Introduction are  $\mathbb{P}$ -almost sure convergences. For objects not related to the diffusion  $X$  we also use the natural notation  $\mathbb{P}$  for a probability.

If  $Z$  is a random variable, its law is denoted by  $\mathcal{L}(Z)$  and if  $A$  is an event of positive probability,  $\mathcal{L}(Z|A)$  denotes the law of  $Z$  conditionally to the event  $A$ .

For  $Z$  an increasing càd-làg process and  $s \geq 0$ , we put respectively  $Z(s-)$ ,  $Z^\natural(s)$  and  $Z^{-1}(s)$  for respectively the left-limit of  $Z$  at  $s$ , the largest jump of  $Z$  before  $s$  and the generalized inverse of  $Z$  at  $s$  :

$$Z(s-) = \lim_{r \nearrow s} Z(r), \quad Z^\natural(s) := \sup_{0 \leq r \leq s} (Z(r) - Z(r-)), \quad Z^{-1}(s) := \inf\{u \geq 0, Z(u) > s\}.$$

For two quantities  $a$  and  $b$  depending on a parameter,  $a \approx b$  means that  $\log(a) \sim \log(b)$  when the parameter converges (generally to 0 or infinity).

## 2. ALMOST SURE BEHAVIOR WHEN $0 < \kappa < 1$

In all this section we assume the hypotheses of Theorems 1.4 and 1.9 :  $0 < \kappa < 1$ ,  $V$  has unbounded variation and there exists  $p > 1$  such that  $V(1) \in L^p$ . The hypothesis of unbounded variation is necessary to approximate the law of the left part of a valley by the law of  $\hat{V}^\uparrow$  and the hypothesis about moments for  $V(1)$  allows to neglect the local time outside the bottom of the valleys. For these reasons, many results of [14] (that are recalled in the next subsection) have been proved under these hypotheses.

**2.1. Traps for the diffusion.** We now recall some definitions about valleys and describe how the diffusion gets trapped into successive valleys. The facts and lemmas stated in this subsection are more or less classical and there are all proved or justified in Subsection 4.3 (except Fact 2.3 which is readily Lemma 3.5 of [14]).

We first recall the notion of  $h$ -extrema. For  $h > 0$ , we say that  $x \in \mathbb{R}$  is an  $h$ -minimum for  $V$  if there exist  $u < x < v$  such that  $V(y) \wedge V(y-) \geq V(x) \wedge V(x-)$  for all  $y \in [u, v]$ ,

$V(u) \geq (V(x) \wedge V(x-)) + h$  and  $V(v-) \geq (V(x) \wedge V(x-)) + h$ . Moreover,  $x$  is an  $h$ -maximum for  $V$  if  $x$  is an  $h$ -minimum for  $-V$ , and  $x$  is an  $h$ -extremum for  $V$  if it is an  $h$ -maximum or an  $h$ -minimum for  $V$ .

Since  $V$  is not a compound Poisson process, it is known (see Proposition VI.4, in [4]) that it takes pairwise distinct values in its local extrema. Combining this with the fact that  $V$  has almost surely càd-làg paths and drifts to  $-\infty$  without being the opposite of a subordinator, we can check that the set of  $h$ -extrema is discrete, forms a sequence indexed by  $\mathbb{Z}$ , unbounded from below and above, and that the  $h$ -minima and  $h$ -maxima alternate. Let  $\mathcal{V} \subset D(\mathbb{R}, \mathbb{R})$  be the set of the environments  $v$  that satisfy the above properties and that are such that

$$v(x) \xrightarrow{x \rightarrow +\infty} -\infty, \quad v(x) \xrightarrow{x \rightarrow -\infty} +\infty, \quad \int_0^{+\infty} e^{v(x)} dx < +\infty.$$

Note that the path of  $V$  belongs to  $\mathcal{V}$  with probability 1.

We denote respectively by  $(m_i, i \in \mathbb{Z})$  and  $(M_i, i \in \mathbb{Z})$  the increasing sequences of  $h$ -minima and of  $h$ -maxima of  $V$ , such that  $m_0 \leq 0 < m_1$  and  $m_i < M_i < m_{i+1}$  for every  $i \in \mathbb{Z}$ . An  $h$ -valleys is the fragments of the trajectory of  $V$  between two  $h$ -maxima.

The valleys are visited successively by the diffusions. For the size of the valleys to be well adapted with respect to the time scale, we have to make the size of the valleys grow with time  $t$ . We are thus interested in  $h_t$ -valleys where

$$h_t := \log(t) - \phi(t), \quad \text{with } \phi(t) := (\log(\log(t)))^\omega, \quad (2.22)$$

where  $\omega > 1$  will be chosen later in accordance with some other parameters. We also define  $N_t$ , the indice of the largest  $h_t$ -minima visited by  $X$  until time  $t$ ,

$$N_t := \max \left\{ k \in \mathbb{N}, \sup_{0 \leq s \leq t} X(s) \geq m_k \right\}.$$

We need deterministic bounds for the number of visited valleys. We define

$$n_t := \lfloor e^{\kappa(1+\delta)\phi(t)} \rfloor \quad \text{and} \quad \tilde{n}_t := e^{\rho\phi(t)},$$

where we fix  $\rho \in ]0, \kappa/(1+\kappa[$  once and for all in all the paper. The following lemma says that with hight probability,  $\tilde{n}_t \leq N_t \leq n_t$  :

**Lemma 2.1.** *There is a positive constant  $c$  such that for all  $t$  large enough,*

$$\mathbb{P}(N_t \geq n_t) \leq e^{-ch_t} \leq e^{-c\phi(t)}, \quad (2.23)$$

$$\mathbb{P}(N_t \leq \tilde{n}_t) \leq e^{-c\phi(t)}. \quad (2.24)$$

We recall the definition of the standard valleys given in [14]. Their interest is mainly the fact that they are defined via successive stopping time, which make them convenient to use in the calculations, and also the fact that they take in consideration the descending phases between two  $h_t$ -minima.

Let  $\delta > 0$ , small enough so that  $(1+3\delta)\kappa < 1$ , be defined once and for all in the paper. Assume  $t$  is large enough so that  $e^{(1-\delta)\kappa h_t} \geq h_t$ . We define  $\tilde{\tau}_0(h_t) = \tilde{L}_0 := 0$  and recursively for

$i \geq 1$ ,

$$\begin{aligned}\tilde{L}_i^\# &:= \inf\{x > \tilde{L}_{i-1}, V(x) \leq V(\tilde{L}_{i-1}) - e^{(1-\delta)\kappa h_t}\}, \\ \tilde{\tau}_i(h_t) &:= \inf\{x \geq \tilde{L}_i^\#, V(x) - \inf_{[\tilde{L}_i^\#, x]} V = h_t\}, \\ \tilde{m}_i &:= \inf\{x \geq \tilde{L}_i^\#, V(x) = \inf_{[\tilde{L}_i^\#, \tilde{\tau}_i(h_t)]} V\}, \\ \tilde{L}_i &:= \inf\{x > \tilde{\tau}_i(h_t), V(x) \leq h_t/2\}, \\ \tilde{\tau}_i^-(a) &:= \sup\{x < \tilde{m}_i, V(x) - V(\tilde{m}_i) \geq a\}, \forall a \in [0, h_t], \\ \tilde{\tau}_i^+(a) &:= \inf\{x > \tilde{m}_i, V(x) - V(\tilde{m}_i) = a\}, \forall a \in [0, h_t].\end{aligned}$$

These random variables depend on  $h_t$  and therefore on  $t$ , even if this does not appear in the notations. We also define

$$\tilde{V}^{(i)}(x) := V(x) - V(\tilde{m}_i), \quad \forall x \in \mathbb{R}.$$

We call  $i^{th}$  *standard valley* the re-centered truncated potential  $(\tilde{V}^{(i)}(x), \tilde{L}_{i-1} \leq x \leq \tilde{L}_i)$ . The law of the bottom of these valleys is given in Fact 4.7 of Section 4.

Similarly as in [14] we define the *deep bottoms* of the  $j^{th}$  standard valleys to be the interval

$$\mathcal{D}_j := [\tilde{\tau}_j^-(\phi(t)^2), \tilde{\tau}_j^+(\phi(t)^2)].$$

**Remark 2.2.** The random times  $\tilde{L}_i^\#$ ,  $\tilde{\tau}_i(h_t)$ , and  $\tilde{L}_i$  are stopping times. As a consequence, the sequence  $(\tilde{V}^{(i)}(x + \tilde{m}_i), \tilde{L}_{i-1} - \tilde{m}_i \leq x \leq \tilde{L}_i - \tilde{m}_i)_{i \geq 1}$  is iid.

We also have that the sequence  $(\tilde{m}_i)_{i \geq 1}$  of the minima of the standard valleys coincides with the sequence  $(m_i)_{i \geq 1}$  with high probability for a large number of indices. Let

$$\mathcal{V}_t := \{v \in \mathcal{V}, \forall i \in \{1, \dots, n_t\}, m_i = \tilde{m}_i\}.$$

Note from the definition of  $\mathcal{V}$  that the sequences  $(\tilde{m}_i)_{i \geq 1}$  and  $(m_i)_{i \geq 1}$  are always defined for any  $v \in \mathcal{V}$  so in particular the event  $\mathcal{V}_t$  is well defined. We have :

**Fact 2.3.** (Lemma 3.5 of [14])

There is a positive constant  $c$  such that for all  $t$  large enough,

$$P(V \in \mathcal{V}_t) \geq 1 - e^{-ch_t}.$$

We define  $X_{\tilde{m}_j} := X(\cdot + \tilde{m}_j)$  which is, according to the Markov property, a diffusion in potential  $V$  starting from  $\tilde{m}_j$ . We also define, for any  $r \in \mathbb{R}$ ,  $H_{X_{\tilde{m}_j}}(r)$  to be the hitting time of  $r$  by  $X_{\tilde{m}_j}$ . When we deal with  $X_{\tilde{m}_j}$  we often need the notation  $A^j(x) = \int_{\tilde{m}_j}^x e^{\tilde{V}^{(j)}(s)} ds$ .

As in [2] and [14], we approximate the repartition function of the renormalized local time by repartition functions of functionals of the sequence  $(e_j S_j^t, e_j S_j^t R_j^t)_{j \geq 1}$  where

$$e_j := \mathcal{L}_X(H(\tilde{L}_j), \tilde{m}_j) / A^j(\tilde{L}_j), \quad S_j^t := \int_{\tilde{\tau}_j^+(h_t/2)}^{\tilde{L}_j} e^{\tilde{V}^{(j)}(u)} du, \quad R_j^t := \int_{\tilde{\tau}_j^-(h_t/2)}^{\tilde{\tau}_j^+(h_t/2)} e^{-\tilde{V}^{(j)}(u)} du.$$

In [14], it is shown that  $e_j$  follows an exponential distribution with parameter  $1/2$  (since the distribution of  $e_j$  does not depend on  $t$  we omit the dependence in  $t$  for  $e_j$  in the notations) and that the random variables  $e_j, S_j^t, R_j^t$ ,  $j \geq 1$  are mutually independent. To simplify notations we define, as in [2] and [14], the process of the renormalized sum of the contributions :

$$\forall s \geq 0, (Y_1^t, Y_2^t)(s) := \frac{1}{t} \sum_{j=1}^{\lfloor se^{\kappa\phi(t)} \rfloor} (e_j S_j^t, e_j S_j^t R_j^t),$$

and the overshoots of  $\sum_{i=1}^{\cdot} e_i S_i^t R_i^t$  : for any  $a \geq 0$ , let us define

$$\mathcal{N}_a := \min \left\{ j \geq 0, \sum_{i=1}^j e_i S_i^t R_i^t > a \right\}.$$

We have

**Fact 2.4.** (*Proposition 4.2 of [14]*)

$(Y_1^t, Y_2^t)$  converges in distribution in  $(D([0, +\infty[, \mathbb{R}^2), J_1)$  to a non-trivial bidimensional subordinator  $(\mathcal{Y}_1, \mathcal{Y}_2)$ .

Let us define some events that happen with high probability. They describe the behavior of the diffusion and provide effective approximations for the time and the local time, in the study of the case  $0 < \kappa < 1$ . On these events, the diffusion leaves the valleys from the right and never goes back to a previous valley, the local time and the time spent by the diffusion are negligible, compared with  $t$ , outside the bottom of the valleys, the supremum of the local time and the time spent by the diffusion in the bottom of the valleys are approximated by the *iid* sequence  $(e_j S_j^t, e_j S_j^t R_j^t)_{j \geq 1}$  :

$$\begin{aligned} \mathcal{E}_t^1 &:= \bigcap_{j=1}^{n_t} \left\{ H_{X_{\tilde{m}_j}}(\tilde{L}_j) < H_{X_{\tilde{m}_j}}(\tilde{L}_{j-1}), H_{X_{\tilde{L}_j}}(+\infty) < H_{X_{\tilde{L}_j}}(\tilde{\tau}_j(h_t)) \right\}, \\ \mathcal{E}_t^2 &:= \bigcap_{j=0}^{n_t-1} \left\{ \sup_{y \in \mathbb{R}} \left( \mathcal{L}_X(H(\tilde{m}_{j+1}), y) - \mathcal{L}_X(H(\tilde{L}_j), y) \right) \leq t e^{(\kappa(1+3\delta)-1)\phi(t)} \right\}, \\ \mathcal{E}_t^3 &:= \bigcap_{j=1}^{n_t} \left\{ \sup_{y \in [\tilde{L}_{j-1}, \tilde{L}_j] \cap \overline{\mathcal{D}_j}} \left( \mathcal{L}_X(H(\tilde{L}_j), y) - \mathcal{L}_X(H(\tilde{m}_j), y) \right) \leq t e^{-2\phi(t)} \right\}, \\ \mathcal{E}_t^4 &:= \bigcap_{j=1}^{n_t} \left\{ \left| \sup_{y \in \mathcal{D}_j} \left( \mathcal{L}_X(H(\tilde{L}_j), y) - \mathcal{L}_X(H(\tilde{m}_j), y) \right) \right| \leq (1 + e^{-\tilde{c}h_t}) \mathcal{L}_X(H(\tilde{L}_j), \tilde{m}_j) \right\}, \\ \mathcal{E}_t^5 &:= \bigcap_{j=1}^{n_t} \left\{ 0 \leq H(\tilde{m}_j) - \sum_{i=1}^{j-1} (H(\tilde{L}_i) - H(\tilde{m}_i)) \leq \frac{2t}{\log h_t} \right\}, \\ \mathcal{E}_t^6 &:= \bigcap_{j=1}^{n_t} \left\{ (1 - e^{-\tilde{c}h_t}) e_j S_j^t \leq \mathcal{L}_X(\tilde{m}_j, H(\tilde{L}_j)) \leq (1 + e^{-\tilde{c}h_t}) e_j S_j^t \right\}, \\ \mathcal{E}_t^7 &:= \bigcap_{j=1}^{n_t} \left\{ (1 - e^{-\tilde{c}h_t}) e_j S_j^t R_j^t \leq H(\tilde{L}_j) - H(\tilde{m}_j) \leq (1 + e^{-\tilde{c}h_t}) e_j S_j^t R_j^t \right\}. \end{aligned}$$

where  $\tilde{c}$  is a fixed positive constant that has been chosen small enough (the constraints for the choice of  $\tilde{c}$  will be precised in the proofs of Fact 2.5 and Fact 2.7). Note that the above events depend both on the environment  $V$  and the brownian motion driving the diffusion.

**Fact 2.5.** *There is a positive constant  $L$  such that for all  $t$  large enough,*

$$\mathbb{P}(\overline{\mathcal{E}_t^7}) \leq e^{-Lh_t}, \quad \sum_{i=1}^7 \mathbb{P}(\overline{\mathcal{E}_t^i}) \leq e^{-L\phi(t)}. \quad (2.25)$$

Fix  $\eta \in ]0, 1[$ . If  $t$  is so large such that  $(1 - e^{-\tilde{c}h_t})^{-1} < (1 + \eta)$  and  $(1 + e^{-\tilde{c}h_t})^{-1}(1 - 2/\log(h_t)) \geq (1 - \eta)$ , then

$$\{V \in \mathcal{V}_t\} \cap \{N_t < n_t\} \cap \mathcal{E}_t^5 \cap \mathcal{E}_t^7 \subset \{\mathcal{N}_{(1-\eta)t} \leq N_t \leq \mathcal{N}_{(1+\eta)t}\}. \quad (2.26)$$

Our proofs are based on the study of the asymptotic of the quantities  $\mathbb{P}(\mathcal{L}_X^*(t) \geq tx_t)$  and  $\mathbb{P}(\mathcal{L}_X^*(t) \leq t/x_t)$  where  $x_t$  depends on  $t$  and goes to infinity with  $t$ . More precisely we define  $x_t$  such that

$$x_t \underset{t \rightarrow +\infty}{\sim} D(\log(\log(t)))^{\mu-1}, \quad (2.27)$$

where  $D > 0$  and  $\mu \in ]1, 2]$ . Precise choices of  $x_t$  will be made later, they will all satisfy (2.27) for some  $D > 0$  and  $\mu \in ]1, 2]$ .

Fix  $\epsilon$  small enough so that Fact 4.8 of Section 4 is satisfied. We define  $\mathcal{G}_t$  to be the set of "good environments" in the following sens :  $v \in \mathcal{V}_t$  belongs to  $\mathcal{G}_t$  if it satisfies the following conditions :

$$\forall j \in \{1, \dots, n_t\}, e^{-\epsilon h_t/4} \leq R_j^t = \int_{\tilde{\tau}_j^-(h_t/2)}^{\tilde{\tau}_j^+(h_t/2)} e^{-(v(u)-v(\tilde{m}_j))} du \leq e^{h_t/8}, \quad (2.28)$$

$$\forall j \in \{1, \dots, n_t\}, \left| A^j(\tilde{\tau}_j^-(h_t/2)) \right| = \left| \int_{\tilde{m}_j}^{\tilde{\tau}_j^-(h_t/2)} e^{-(v(u)-v(\tilde{m}_j))} du \right| \leq e^{5h_t/8}, \quad (2.29)$$

$$\forall j \in \{1, \dots, n_t\}, A^j(\tilde{\tau}_j^+(h_t/2)) = \int_{\tilde{m}_j}^{\tilde{\tau}_j^+(h_t/2)} e^{-(v(u)-v(\tilde{m}_j))} du \leq e^{5h_t/8}, \quad (2.30)$$

$$\forall j \in \{1, \dots, n_t\}, \int_{\tilde{L}_{j-1}}^{\tilde{\tau}_j^-(h_t/2)} e^{-(v(u)-v(\tilde{m}_j))} du \leq e^{-\epsilon h_t}, \quad (2.31)$$

$$\forall j \in \{1, \dots, n_t\}, \int_{\tilde{\tau}_j^+(h_t/2)}^{\tilde{L}_j} e^{-(v(u)-v(\tilde{m}_j))} du \leq e^{-\epsilon h_t}, \quad (2.32)$$

$$P^v(\cup_{i=1}^7 \overline{\mathcal{E}}_t^i) \leq e^{-L\phi(t)/2}. \quad (2.33)$$

where  $P^v(\cdot)$  is defined in Subsection 1.3 and  $L$  is the constant defined in Fact 2.5. Note that, since  $\mathcal{G}_t \subset \mathcal{V}_t$ , we will often use the fact that  $\tilde{m}_j = m_j$  for  $v \in \mathcal{G}_t$  and  $j \leq n_t$ .

**Lemma 2.6.** *There is a positive constant  $c$  such that for all  $t$  large enough,*

$$P(V \in \mathcal{G}_t) \geq 1 - e^{-c\phi(t)}.$$

We need that, as in Lemma 5.3 of [2], an inequality for the local time in the bottom of a valley is related to an inequality for the corresponding random variable  $R_k^t$ . Since we deal with unlikely inequalities for the local time, we have to prove the negligibility of the event where such inequalities happen but not the corresponding inequalities for  $R_k^t$ . Recall the notations  $X_{\tilde{m}_j}$  and  $H_{X_{\tilde{m}_j}}(\cdot)$  defined above. For any fixed environment  $v \in \mathcal{G}_t$  and  $z \in [0, 1]$  we define

$$\begin{aligned} \mathcal{E}_t^8(v, k, z) &:= \left\{ (1-z) \frac{(1 + e^{-\tilde{c}h_t})}{(1 - e^{-\tilde{c}h_t})} < x_t R_k^t, \sup_{\mathcal{D}_k} \mathcal{L}_{X_{\tilde{m}_k}}(t(1-z), \cdot) \geq tx_t, H_{X_{\tilde{m}_k}}(\tilde{L}_k) < H_{X_{\tilde{m}_k}}(\tilde{L}_{k-1}) \right\}, \\ \mathcal{E}_t^9(v, k, z) &:= \left\{ R_k^t/x_t < (1 - e^{-\tilde{c}h_t})(1-z), \mathcal{L}_{X_{\tilde{m}_k}}(t(1-z), \tilde{m}_k) \leq t/x_t, \right. \\ &\quad \left. H_{X_{\tilde{m}_k}}(\tilde{L}_k) \geq t(1-z), H_{X_{\tilde{m}_k}}(\tilde{L}_k) < H_{X_{\tilde{m}_k}}(\tilde{L}_{k-1}) \right\}, \end{aligned}$$

where  $\tilde{c}$  is the same as in the definitions of  $\mathcal{E}_t^4, \mathcal{E}_t^6$  and  $\mathcal{E}_t^7$ . Note that the inequality  $(1-z)(1 + e^{-\tilde{c}h_t})/(1 - e^{-\tilde{c}h_t}) < x_t R_k^t$  (resp.  $R_k^t/x_t < (1 - e^{-\tilde{c}h_t})(1-z)$ ) only depends on the environment  $v$ . We consider  $\mathcal{E}_t^8(v, k, z)$  (resp.  $\mathcal{E}_t^9(v, k, z)$ ) to equal  $\emptyset$  when  $v$  is such that this equality is not satisfied. In the rest of the paper, we use the same convention when we consider events, at fixed environment, that partially depend on the environment.

**Fact 2.7.** *There is a positive constant  $c$  such that for  $t$  large enough and any fixed environment  $v \in \mathcal{G}_t$  we have*

$$P^v \left( \bigcup_{k=1}^{n_t} \{N_t \geq k, H(\tilde{m}_k)/t \leq 1 - 4/\log(h_t)\} \cap \mathcal{E}_t^8(v, k, H(\tilde{m}_k)/t) \right) \leq e^{-c\phi(t)}, \quad (2.34)$$

$$P^v \left( \bigcup_{k=1}^{n_t} \mathcal{E}_t^8(v, k, 1 - 4/\log(h_t)) \right) \leq e^{-c\phi(t)}, \quad (2.35)$$

$$P^v \left( \bigcup_{k=1}^{n_t} \{N_t \geq k\} \cap \mathcal{E}_t^9(v, k, H(\tilde{m}_k)/t) \right) \leq e^{-c\phi(t)}. \quad (2.36)$$

Note that in the above fact,  $v$  is fixed in  $\mathcal{G}_t \subset \mathcal{V}_t$  so  $N_t \geq k$  implies  $H(\tilde{m}_k)/t \leq 1$ . The quantities in the above fact are thus well defined.

We know from Fact 2.5 that the contributions to the local time and to the time spent of the successive valleys are approximated by the *iid* sequence  $(e_j S_j^t, e_j S_j^t R_j^t)_{j \geq 1}$ . Since we deal with extreme values of the local time, we need to know the right tail of the distributions of  $e_j S_j^t$  and of  $e_j S_j^t R_j^t$ . We also need informations about the extreme values of  $R_j^t$ . All these are given by the next fact from [14] :

**Fact 2.8.** *Fix  $\eta \in ]0, 1/3[$  and let  $C'$  be the constant in Lemma 4.15 of [14]. We have*

$$\lim_{t \rightarrow +\infty} \sup_{x \in [e^{-(1-2\eta)\phi(t)}, +\infty[} \left| x^\kappa e^{\kappa\phi(t)} \mathbb{P}(e_1 S_1^t/t > x) - C' \right| = 0, \quad (2.37)$$

$$\lim_{t \rightarrow +\infty} \sup_{y \in [e^{-(1-3\eta)\phi(t)}, +\infty[} \left| y^\kappa e^{\kappa\phi(t)} \mathbb{P}(e_1 S_1^t R_1^t/t > y) - C' \mathbb{E}[\mathcal{R}^\kappa] \right| = 0. \quad (2.38)$$

$(R_1^t)_{t>0}$  converges in distribution to  $\mathcal{R}$  and there exists a positive  $\lambda_0$  such that

$$\forall \lambda < \lambda_0, \quad \mathbb{E} \left[ e^{\lambda R_1^t} \right] \xrightarrow{t \rightarrow +\infty} \mathbb{E} \left[ e^{\lambda \mathcal{R}} \right], \quad (2.39)$$

where the above quantities are all finite. This entails the convergence of the moments of any positive order of  $R_1^t$  to those of  $\mathcal{R}$  when  $t$  goes to infinity.

Finally, let us state a general lemma about the diffusion  $X$  :

**Lemma 2.9.** *Let  $q$  be the constant in Theorem 1.4 of [14]. There is a positive constant  $c$  such that for all  $r$  and  $t$  large enough,*

$$\mathbb{P} \left( \sup_{[0,t]} X \geq 2t^\kappa e^{\kappa\delta(\log(\log(t)))^\omega} / q \right) \leq e^{-ct}, \quad (2.40)$$

$$\mathbb{P} \left( X(t) \leq t^\kappa e^{(\rho-\kappa)(\log(\log(t)))^\omega} / 2q \right) \leq e^{-c\phi(t)}, \quad (2.41)$$

$$\mathbb{P} \left( \inf_{[0,+\infty[} X \leq -r \right) \leq 3r^{-1}, \quad (2.42)$$

$$\mathbb{P} \left( \inf_{]-\infty, 0]} \mathcal{L}_X(+\infty, \cdot) > r \right) \leq 3r^{-\kappa/(2+\kappa)}. \quad (2.43)$$

**2.2. Decomposition of the diffusion into independent parts.** An important point in our proofs is to give a decomposition of the trajectory of  $X$  that makes independence appear in order to apply the Borel-Cantelli Lemma. Let us fix  $a > 1$  and define the sequences  $t_n := e^{n^a}$ ,  $u_n := e^{\kappa(n^a - 2an^{a-1}/3)}$ ,  $v_n := e^{\kappa(n^a - an^{a-1}/3)}$ . Let  $X^n := X(H(v_n) + \cdot)$ , the diffusion shifted by the hitting time of  $v_n$  and  $T_n := \min\{t_n, \tau(X^n, u_n), \tau(X^n, u_{n+1})\}$ . Note that from the Markov property for  $X$  at time  $H(v_n)$  and the stationarity of the increments of  $V$ ,  $X^n - v_n$  is equal in

law to  $X$  under the annealed probability  $\mathbb{P}$ . Let  $n_0$  be large enough so that  $u_n \leq v_n \leq u_{n+1}$  for all  $n \geq n_0$ . We define the events

$$\mathcal{C}_n := \{T_n = t_n\} \quad \text{and} \quad \mathcal{D}_n := \{H(v_n) < t_n/n\}.$$

The idea is that the sequence of processes  $(X^n(t), 0 \leq t \leq T_n)_{n \geq n_0}$  is independent so, intersecting a sequence  $(\mathcal{B}_n)_{n \geq n_0}$  of interesting events (where each event  $\mathcal{B}_n$  only depends on  $(X^n(t), 0 \leq t \leq t_n)$ ) with  $\mathcal{C}_n$  will result in  $(\mathcal{B}_n \cap \mathcal{C}_n)_{n \geq n_0}$ , a sequence of independent events. Since  $X^n$  is  $X$  shifted by  $H(v_n)$ , the event  $\mathcal{D}_n$  is useful to neglect this time shift when dealing with the renormalization of the local time. We will need the following lemma :

**Lemma 2.10.**

$$\sum_{n \geq 1} \mathbb{P}(\overline{\mathcal{C}_n}) < +\infty, \quad (2.44)$$

and

$$\sum_{n \geq 1} \mathbb{P}(\overline{\mathcal{D}_n}) < +\infty. \quad (2.45)$$

*Proof.* Here again, let  $q$  be the constant in Theorem 1.4 of [14]. First, notice from the definitions of  $t_n$ ,  $u_n$  and  $v_n$  that for all  $n$  large enough,

$$2t_n^\kappa e^{\kappa \delta(\log(\log(t_n)))^\omega} / q < u_{n+1} - v_n, \quad (2.46)$$

and

$$(t_n/n)^\kappa e^{(\rho-\kappa)(\log(\log(t_n/n)))^\omega} / 2q > v_n. \quad (2.47)$$

From the definition of  $\mathcal{C}_n$  and the Markov property applied to  $X$  at time  $H(v_n)$  we have

$$\begin{aligned} \mathbb{P}(\overline{\mathcal{C}_n}) &\leq \mathbb{P}(\tau(X^n, u_{n+1}) < t_n) + \mathbb{P}(\tau(X^n, u_n) < +\infty) \\ &\leq \mathbb{P}(\tau(X, u_{n+1} - v_n) < t_n) + \mathbb{P}(\tau(X, u_n - v_n) < +\infty) \\ &\leq \mathbb{P}\left(\sup_{[0, t_n]} X \geq u_{n+1} - v_n\right) + \mathbb{P}\left(\inf_{[0, +\infty[} X \leq u_n - v_n\right) \\ &\leq e^{-ch_{t_n}} + 3/(v_n - u_n) \end{aligned}$$

where  $c$  is the constant in Lemma 2.9. The last inequality is true for  $n$  large enough and comes, for the first term, from the combination of (2.46) and (2.40), and, for the second term, from (2.42). Recall that  $e^{-h_{t_n}} \approx e^{-\log(t_n)} = e^{-n^a}$ . We thus deduce (2.44).

From the definition of  $\mathcal{D}_n$ , (2.47) and (2.41) we have for all  $n$  large enough,

$$\mathbb{P}(\overline{\mathcal{D}_n}) \leq \mathbb{P}(X(t_n/n) \leq v_n) \leq \mathbb{P}\left(X(t_n/n) \leq (t_n/n)^\kappa e^{(\rho-\kappa)(\log(\log(t_n/n)))^\omega} / 2q\right) \leq e^{-c\phi(t_n/n)},$$

where  $c$  is the constant in Lemma 2.9. Since  $e^{-c\phi(t_n/n)} = e^{-c(\log(\log(t_n/n)))^\omega} \approx e^{-ca^\omega(\log(n))^\omega}$  we deduce (2.45). □

**2.3. The limsup.** We study the asymptotic of the quantity  $\mathbb{P}(\mathcal{L}_X^*(t)/t \geq x_t)$ . Recall that  $x_t$  is defined in (2.27) where  $D > 0$  and  $\mu \in ]1, 2]$  are fixed constants. In all this subsection the parameter  $\omega$  in (2.22) is fixed in  $]1, \mu[$ . We have :



**Proposition 2.11.** *There is a positive constant  $c$  such that for all  $a > 1$  and  $t$  large enough we have,*

$$\begin{aligned} \mathbb{P}\left(Y_1^{\natural,t}\left(Y_2^{-1,t}(1/a)-\right) \geq a x_t\right) - e^{-c\phi(t)} &\leq \mathbb{P}\left(\mathcal{L}_X^*(t) \geq tx_t\right) \\ &\leq \mathbb{P}\left(Y_1^{\natural,t}\left(Y_2^{-1,t}(a)-\right) \geq x_t/a\right) + \mathbb{P}\left(R_1^t \leq \frac{a}{x_t}\right) + e^{-c\phi(t)}. \end{aligned}$$

Note that the functional of  $(Y_1^t, Y_2^t)$  involved in this proposition is  $Y_1^{\natural,t}(Y_2^{-1,t}(\cdot)-)$  which represents the supremum of the local time before (and not including) the last valley. Even though, as we see in Proposition 4.1 of [14], the repartition function of  $\mathcal{L}_X^*(t)/t$  involves a complex functional of  $(Y_1^t, Y_2^t)$  that represents the last valley (together with the functional  $Y_1^{\natural,t}(Y_2^{-1,t}(\cdot))$  that represents the previous valleys), Proposition 2.11 says that the right tail of this distribution function does not involve the last valley. Before proving this proposition we prove some lemmas.

**Lemma 2.12.** *There is a positive constant  $C$  such that for  $t$  large enough,*

$$\sum_{k=1}^{n_t} \mathbb{P}(H(\tilde{m}_k)/t \leq 1) \leq Ce^{\kappa\phi(t)}.$$

*Proof.* Since for all  $k \geq 1$  we have almost surely  $H(\tilde{m}_k) \geq \sum_{j=1}^{k-1} H(\tilde{L}_j) - H(\tilde{m}_j)$ , we deduce that

$$\begin{aligned} \sum_{k=1}^{n_t} \mathbb{P}(H(\tilde{m}_k)/t \leq 1) &\leq \sum_{k=1}^{n_t} \mathbb{P}\left(\sum_{j=1}^{k-1} H(\tilde{L}_j) - H(\tilde{m}_j)/t \leq 1\right) \\ &\leq \sum_{k=1}^{n_t} \mathbb{P}\left(\sum_{j=1}^{k-1} (1 - e^{-\tilde{c}h_t}) e_j S_j^t R_j^t/t \leq 1\right) + n_t \mathbb{P}(\overline{\mathcal{E}}_t^7) \\ &\leq e^{(1-e^{-\tilde{c}h_t})^{-1}} \sum_{k=1}^{n_t} \mathbb{E}\left[e^{-\sum_{j=1}^{k-1} e_j S_j^t R_j^t/t}\right] + n_t e^{-Lh_t} \\ &= e^{(1-e^{-\tilde{c}h_t})^{-1}} \sum_{k=1}^{n_t} \left(\mathbb{E}\left[e^{-e_1 S_1^t R_1^t/t}\right]\right)^{k-1} + e^{(1+\delta)\phi(t)-Lh_t}, \end{aligned}$$

where we used the definition of  $\mathcal{E}_t^7$ , Markov's inequality, (2.25), the fact that the sequence  $(e_j S_j^t R_j^t)_{j \geq 1}$  is *iid* and the definition of  $n_t$ . For  $t$  large enough we thus get

$$\begin{aligned} \sum_{k=1}^{n_t} \mathbb{P}(H(\tilde{m}_k)/t \leq 1) &\leq e^2 \sum_{k=1}^{+\infty} \left(\mathbb{E}\left[e^{-e_1 S_1^t R_1^t/t}\right]\right)^{k-1} + e^{-Lh_t/2} \\ &= \frac{e^2}{1 - \mathbb{E}\left[e^{-e_1 S_1^t R_1^t/t}\right]} + e^{-Lh_t/2}. \end{aligned} \tag{2.48}$$

Then,

$$\begin{aligned} 1 - \mathbb{E}\left[e^{-e_1 S_1^t R_1^t/t}\right] &= \int_0^{+\infty} e^{-u} \mathbb{P}(e_1 S_1^t R_1^t/t > u) du \\ &\geq \int_{e^{-\phi(t)/2}}^{+\infty} e^{-u} \mathbb{P}(e_1 S_1^t R_1^t/t > u) du \\ &= e^{-\kappa\phi(t)} \int_{e^{-\phi(t)/2}}^{+\infty} u^{-\kappa} e^{-u} \left(u^{\kappa} e^{\kappa\phi(t)} \mathbb{P}(e_1 S_1^t R_1^t/t > u)\right) du. \end{aligned}$$

We can now use (2.38) with  $\eta = 1/6$  and get that for all  $t$  large enough

$$1 - \mathbb{E} \left[ e^{-e_1 S_1^t R_1^t / t} \right] \geq \frac{\mathcal{C}' \mathbb{E}[\mathcal{R}^\kappa]}{2} e^{-\kappa \phi(t)} \int_{e^{-\phi(t)/2}}^{+\infty} u^{-\kappa} e^{-u} du \underset{t \rightarrow +\infty}{\sim} \frac{\mathcal{C}' \mathbb{E}[\mathcal{R}^\kappa]}{2} e^{-\kappa \phi(t)} \int_0^{+\infty} u^{-\kappa} e^{-u} du.$$

Putting into (2.48), we get the result for  $t$  large enough.  $\square$

We now link the asymptotic of  $\mathbb{P}(R_1^t \leq a/x_t)$  with the left tail of  $I(V^\uparrow)$ . We have to make a distinction between the case where  $V$  possesses negative jumps and the case where  $V$  possesses no negative jumps, that is,  $V$  is the  $\kappa$ -drifted brownian motion.  $R_1^t$  is an exponential functional of the bottom of the first valley. In the first case, due to the jumps, the left side of the bottom of the valley can be neglected, so only the right side counts. In the second case, both sides have the same law, so both have to be taken in consideration in the left tail of  $R_1^t$ .

**Lemma 2.13.** *Let  $z_t$  go to infinity with  $t$  satisfying  $(\log(z_t))^2 \ll h_t$ . Assume  $V$  possesses negative jumps. There is a positive constant  $c$  such that for any  $a > 1$  and  $t$  large enough,*

$$e^{-c(\log((1-1/a)/z_t))^2} \mathbb{P} \left( I(V^\uparrow) \leq 1/a z_t \right) \leq \mathbb{P} (R_1^t \leq 1/z_t) \leq 2 \mathbb{P} \left( I(V^\uparrow) \leq a/z_t \right). \quad (2.49)$$

If  $V := W_\kappa$ , the  $\kappa$ -drifted brownian motion then for  $t$  large enough,

$$\mathbb{P} (\mathcal{R} \leq 1/z_t) - 2e^{-\delta \kappa h_t/3} \leq \mathbb{P} (R_1^t \leq 1/z_t) \leq 2 \mathbb{P} (\mathcal{R} \leq a/z_t) + 2e^{-\delta \kappa h_t/3}. \quad (2.50)$$

*Proof.* We first assume that  $V$  possesses negative jumps. Recall that  $R_1^t = \int_{\tilde{\tau}_1^-(h_t/2)}^{\tilde{\tau}_1^+(h_t/2)} e^{-\tilde{V}^{(1)}(u)} du$  so, using the equality in law between  $(\tilde{V}^{(i)}(\tilde{m}_i + x), 0 \leq x \leq \tilde{\tau}_i(h) - \tilde{m}_i)$  and  $(V^\uparrow(x), 0 \leq x \leq \tau(V^\uparrow, h))$  given by Fact 4.7, we get

$$R_1^t \geq \int_{\tilde{m}_1}^{\tilde{\tau}_1^+(h_t/2)} e^{-\tilde{V}^{(1)}(u)} du \stackrel{\mathcal{L}}{=} \int_0^{\tau(V^\uparrow, h_t/2)} e^{-V^\uparrow(u)} du,$$

so

$$\mathbb{P} (R_1^t \leq 1/z_t) \leq \mathbb{P} \left( \int_0^{\tau(V^\uparrow, h_t/2)} e^{-V^\uparrow(u)} du \leq 1/z_t \right). \quad (2.51)$$

According to Lemma 4.6, for some positive constant  $c$  and  $t$  large enough,  $(1 - e^{-ch_t}) \times \mathbb{P} \left( \int_0^{\tau(V^\uparrow, h_t/2)} e^{-V^\uparrow(u)} du \leq 1/z_t \right)$  is less than

$$\begin{aligned} \mathbb{P} \left( \int_0^{\tau(V^\uparrow, h_t/2)} e^{-V^\uparrow(u)} du \leq 1/z_t \right) &\times \mathbb{P} \left( \int_{\tau(V^\uparrow, h_t/2)}^{+\infty} e^{-V^\uparrow(u)} du \leq e^{-h_t/4} \right) \\ &\leq \mathbb{P} \left( I(V^\uparrow) \leq 1/z_t + e^{-h_t/4} \right). \end{aligned} \quad (2.52)$$

Combining with (2.51) we get that for  $t$  large enough,

$$\mathbb{P} (R_1^t \leq 1/z_t) \leq 2 \mathbb{P} \left( I(V^\uparrow) \leq 1/z_t + e^{-h_t/4} \right) \leq 2 \mathbb{P} \left( I(V^\uparrow) \leq a/z_t \right),$$

because  $e^{-h_t/4} \leq (a-1)/z_t$  for large  $t$  (thanks to the hypothesis  $(\log(z_t))^2 \ll h_t$ ). This is the asserted upper bound in (2.49).

On the other hand, using the independence between the left and right parts of the valleys (given by Fact 4.7), we get

$$\mathbb{P}(R_1^t \leq 1/z_t) \geq \mathbb{P}\left(\int_{\tilde{m}_1}^{\tilde{\tau}_1^+(h_t/2)} e^{-\tilde{V}^{(1)}(u)} du \leq 1/az_t\right) \times \mathbb{P}\left(\int_{\tilde{\tau}_1^-(h_t/2)}^{\tilde{m}_1} e^{-\tilde{V}^{(1)}(u)} du \leq (1-1/a)/z_t\right) \quad (2.53)$$

From Fact 4.7, we get that the first factor equals

$$\mathbb{P}\left(\int_0^{\tau(V^\uparrow, h_t/2)} e^{-V^\uparrow(u)} du \leq 1/az_t\right) \geq \mathbb{P}\left(I(V^\uparrow) \leq 1/az_t\right),$$

while the second factor is more than

$$\begin{aligned} & \mathbb{E}\left[\frac{c_{h_t}}{1 - e^{-\kappa \hat{V}^\uparrow(\tau(\hat{V}^\uparrow, h_t/2))}}; \int_0^{\tau(\hat{V}^\uparrow, h_t/2)} e^{-\hat{V}^\uparrow(u)} du \leq (1-1/a)/z_t\right] - 2e^{-\delta\kappa h_t/3} \\ & \geq c_{h_t} \mathbb{P}\left(\int_0^{+\infty} e^{-\hat{V}^\uparrow(u)} du \leq (1-1/a)/z_t\right) - 2e^{-\delta\kappa h_t/3}. \end{aligned}$$

Then,  $c_{h_t} \geq c_1$  when  $h_t \geq 1$  so, putting in (2.53), we get that for  $t$  large enough,

$$\mathbb{P}(R_1^t \leq 1/z_t) \geq \mathbb{P}\left(I(V^\uparrow) \leq 1/az_t\right) \times \left(c_1 \mathbb{P}\left(I(\hat{V}^\uparrow) \leq (1-1/a)/z_t\right) - 2e^{-\delta\kappa h_t/3}\right). \quad (2.54)$$

According to the combination of Proposition 1.15 and Theorem 1.16 of [13], there is a positive constant  $c$  such that for  $t$  large enough,

$$\mathbb{P}\left(I(\hat{V}^\uparrow) \leq (1-1/a)/z_t\right) \geq e^{-c(\log((1-1/a)/z_t))^2}.$$

Thanks to the hypothesis  $(\log(z_t))^2 \ll h_t$  we deduce that, for  $c$  decreased a little, the second factor in the right hand side of (2.54) is more than  $e^{-c(\log((1-1/a)/z_t))^2}$ . This yields the lower bound in (2.49).

We now consider the case where  $V$  is the  $\kappa$ -drifted brownian motion  $W_\kappa$ . Let  $Z_1$  and  $Z_2$  be two independent versions of the process  $W_\kappa^\uparrow$ . Since  $W_\kappa$  has no jumps, the density of the process  $P^{(2)}$  in Fact 4.7 is almost surely constant so  $P^{(2)}$  is equal in law to  $(\hat{W}_\kappa^\uparrow(x), 0 \leq x \leq \tau(\hat{W}_\kappa^\uparrow, h_t/2)) = (W_\kappa^\uparrow(x), 0 \leq x \leq \tau(W_\kappa^\uparrow, h_t/2))$  (the last equality comes from the fact that  $\hat{W}_\kappa^\uparrow = W_\kappa^\uparrow$  and  $W_\kappa$  is continuous). Combining this with (4.121) and the equality in law between  $(\tilde{V}^{(i)}(\tilde{m}_i + x), 0 \leq x \leq \tilde{\tau}_i(h) - \tilde{m}_i)$  and  $(V^\uparrow(x), 0 \leq x \leq \tau(V^\uparrow, h))$  (both are from Fact 4.7), we get

$$\mathbb{P}(R_1^t \leq 1/z_t) \leq \mathbb{P}\left(\int_0^{\tau(Z_1, h_t/2)} e^{-Z_1(u)} du + \int_0^{\tau(Z_2, h_t/2)} e^{-Z_2(u)} du \leq 1/z_t\right) + 2e^{-\delta\kappa h_t/3}, \quad (2.55)$$

$$\mathbb{P}(R_1^t \leq 1/z_t) \geq \mathbb{P}\left(\int_0^{\tau(Z_1, h_t/2)} e^{-Z_1(u)} du + \int_0^{\tau(Z_2, h_t/2)} e^{-Z_2(u)} du \leq 1/z_t\right) - 2e^{-\delta\kappa h_t/3}. \quad (2.56)$$

Reasoning as in (2.52) we get for some positive constant  $c$  :

$$\begin{aligned} & (1 - e^{-ch_t})^2 \mathbb{P}\left(\int_0^{\tau(Z_1, h_t/2)} e^{-Z_1(u)} du + \int_0^{\tau(Z_2, h_t/2)} e^{-Z_2(u)} du \leq 1/z_t\right) \\ & \leq \mathbb{P}\left(I(Z_1) + I(Z_2) \leq 1/z_t + 2e^{-h_t/4}\right). \end{aligned}$$

Combining with (2.55) we get that for  $t$  large enough,

$$\mathbb{P}(R_1^t \leq 1/z_t) \leq 2\mathbb{P}(I(Z_1) + I(Z_2) \leq 1/z_t + 2e^{-h_t/4}) + 2e^{-\delta\kappa h_t/3} \leq 2\mathbb{P}(\mathcal{R} \leq a/z_t) + 2e^{-\delta\kappa h_t/3},$$

because  $2e^{-h_t/4} \leq (a-1)/z_t$  for large  $t$  and because, from the definitions of  $\mathcal{R}$ ,  $Z_1$  and  $Z_2$  we have  $\mathcal{R} \stackrel{\mathfrak{I}}{=} I(Z_1) + I(Z_2)$ . The above inequality is the asserted upper bound in (2.50).

On the other hand, we have trivially

$$\mathbb{P}\left(\int_0^{\tau(Z_1, h_t/2)} e^{-Z_1(u)} du + \int_0^{\tau(Z_2, h_t/2)} e^{-Z_2(u)} du \leq 1/z_t\right) \geq \mathbb{P}(I(Z_1) + I(Z_2) \leq 1/z_t),$$

and combining with (2.56) we get that for  $t$  large enough,

$$\mathbb{P}(R_1^t \leq 1/z_t) \geq \mathbb{P}(I(Z_1) + I(Z_2) \leq 1/z_t) - 2e^{-\delta\kappa h_t/3} = \mathbb{P}(\mathcal{R} \leq 1/z_t) - 2e^{-\delta\kappa h_t/3}.$$

This yields the lower bound in (2.50). □

The next lemma studies the contribution of the last valley :

**Lemma 2.14.** *There is a positive constant  $c$  such that for all  $u > 1$  and  $t$  large enough,*

$$\mathbb{P}\left(\sup_{\mathcal{D}_{N_t}} (\mathcal{L}_X(t, \cdot) - \mathcal{L}_X(H(\tilde{m}_{N_t}), \cdot)) \geq tx_t\right) \leq \mathbb{P}\left(R_1^t \leq \frac{u}{x_t}\right) + e^{-c\phi(t)}.$$

*Proof.* We fix  $v \in \mathcal{G}_t$ , a realization of the environment. Let us define

$$\mathcal{E}_t(v, k, z) := \left\{ \sup_{y \in \mathcal{D}_k} \mathcal{L}_{X_{\tilde{m}_k}}(t(1-z), y) \geq tx_t, \right. \\ \left. H_{X_{\tilde{m}_k}}(\tilde{m}_{k+1}) \geq t(1-z), H_{X_{\tilde{m}_k}}(\tilde{L}_k) < H_{X_{\tilde{m}_k}}(\tilde{L}_{k-1}) \right\}.$$

We have

$$\begin{aligned} & P^v \left( \sup_{\mathcal{D}_{N_t}} (\mathcal{L}_X(t, \cdot) - \mathcal{L}_X(H(\tilde{m}_{N_t}), \cdot)) \geq tx_t, N_t < n_t, \mathcal{E}_t^1 \right) \\ & \leq \sum_{k=1}^{n_t} \int_0^1 P^v(\mathcal{E}_t(v, k, z), H(\tilde{m}_k)/t \in dz). \end{aligned} \quad (2.57)$$

The fact that the sum stops at  $n_t$  comes from  $N_t < n_t$  together with the fact that  $v \in \mathcal{G}_t \subset \mathcal{V}_t$ . From the definitions of  $\mathcal{E}_t(v, k, z)$ ,  $\mathcal{E}_t^8(v, k, z)$ ,  $\mathcal{E}_t^5$  and  $\mathcal{E}_t^7$  we have

$$\begin{aligned} \mathcal{E}_t(v, k, z) \subset & \left\{ \frac{1-z}{R_k^t} \frac{(1+e^{-\tilde{c}h_t})}{(1-e^{-\tilde{c}h_t})} \geq x_t, (1+e^{-\tilde{c}h_t})e_k S_k^t R_k^t \geq t(1-z-2/\log(h_t)) \right\} \\ & \cup \mathcal{E}_t^8(v, k, z) \cup \overline{\mathcal{E}_t^5} \cup \overline{\mathcal{E}_t^7}. \end{aligned}$$

When  $z \leq 1 - 4/\log(h_t)$  we have, on the big event in the right hand side,

$$\begin{aligned} t(1-z)/2 & \leq t(1-z-2/\log(h_t)) \leq (1+e^{-\tilde{c}h_t})e_k S_k^t R_k^t \\ & \leq (1+e^{-\tilde{c}h_t})^2 (1-e^{-\tilde{c}h_t})^{-1} (1-z)e_k S_k^t / x_t \\ & \leq 2(1-z)e_k S_k^t / x_t, \end{aligned}$$

for  $t$  large enough, and trivially  $(1-z)(1+e^{-\tilde{c}h_t})(1-e^{-\tilde{c}h_t})^{-1} \leq 1+e^{-\tilde{c}h_t/2}$  for  $t$  large enough. As a consequence, for  $z \in [0, 1-4/\log(h_t)]$  and  $t$  large enough,

$$\mathcal{E}_t(v, k, z) \subset \left\{ R_k^t \leq (1+e^{-\tilde{c}h_t/2})/x_t, e_k S_k^t/t \geq x_t/4 \right\} \cup \mathcal{E}_t^8(v, k, z) \cup \overline{\mathcal{E}_t^5} \cup \overline{\mathcal{E}_t^7}.$$

Note also that the sum in (2.57) corresponds to disjoint events (so it is actually the probability of a union of events). We thus get that  $\sum_{k=1}^{n_t} \int_0^{1-4/\log(h_t)} P^v(\mathcal{E}_t(v, k, z), H(\tilde{m}_k)/t \in dz)$  is less than

$$\begin{aligned} & \sum_{k=1}^{n_t} \int_0^{1-4/\log(h_t)} P^v \left( R_k^t \leq (1+e^{-\tilde{c}h_t/2})/x_t, e_k S_k^t/t \geq x_t/4, H(\tilde{m}_k)/t \in dz \right) \\ & + P^v \left( \overline{\mathcal{E}_t^5} \cup \overline{\mathcal{E}_t^7} \cup \bigcup_{k=1}^{n_t} \{N_t \geq k, H(\tilde{m}_k)/t \leq 1-4/\log(h_t)\} \cap \mathcal{E}_t^8(v, k, H(\tilde{m}_k)/t) \right). \end{aligned}$$

Now, recall that  $(S_k^t, R_k^t)$  only depends on  $v$  and that,  $v$  being fixed,  $e_k$  belongs to the  $\sigma$ -field  $\sigma(X(t), t \geq H(\tilde{m}_k))$ . In other words, it only depends on the diffusion after time  $H(\tilde{m}_k)$ . On the other hand,  $H(\tilde{m}_k)$  is measurable with respect to the  $\sigma$ -field  $\sigma(X(t), 0 \leq t \leq H(\tilde{m}_k))$ . From the Markov property applied to  $X$  at  $H(\tilde{m}_k)$ , we get that  $H(\tilde{m}_k)$  is independent from  $(e_k, S_k^t, R_k^t)$  so the above is less than

$$\begin{aligned} & \sum_{k=1}^{n_t} P^v \left( R_k^t \leq (1+e^{-\tilde{c}h_t/2})/x_t, e_k S_k^t/t \geq x_t/4 \right) \times P^v(H(\tilde{m}_k)/t \leq 1) \\ & + P^v \left( \overline{\mathcal{E}_t^5} \cup \overline{\mathcal{E}_t^7} \cup \bigcup_{k=1}^{n_t} \{N_t \geq k, H(\tilde{m}_k)/t \leq 1-4/\log(h_t)\} \cap \mathcal{E}_t^8(v, k, H(\tilde{m}_k)/t) \right) \\ & \leq \sum_{k=1}^{n_t} P^v \left( R_k^t \leq (1+e^{-\tilde{c}h_t/2})/x_t, e_k S_k^t/t \geq x_t/4 \right) \times P^v \left( H(\tilde{L}_{k-1})/t \leq 1 \right) + e^{-c\phi(t)}, \end{aligned}$$

where  $c$  is a positive constant and where we used the fact that  $\tilde{L}_{k-1} \leq \tilde{m}_k$  for the first term and the fact that  $v \in \mathcal{G}_t$  together with (2.33) and (2.34) for the second term. Now, note that the first factor in the above product only depends on  $(\tilde{v}^{(k)}(x), \tilde{L}_{k-1} \leq x \leq \tilde{L}_k)$  (that is, only on  $v$  shifted at time  $\tilde{L}_{k-1}$ ) while the second factor depends on  $v$  before time  $\tilde{L}_{k-1}$ . This time  $\tilde{L}_{k-1}$  is a stopping time for the Lévy process  $V$  of which  $v$  is a fixed possible path. As a consequence, when we integrate the above inequality with respect to  $v$  over  $D(\mathbb{R}, \mathbb{R}) = \mathcal{G}_t \cup \overline{\mathcal{G}_t}$  equipped with the probability measure  $P$ , we get that the two factors are independent so  $E[\sum_{k=1}^{n_t} \int_0^{1-4/\log(h_t)} P^V(\mathcal{E}_t(V, k, z), H(\tilde{m}_k)/t \in dz)]$ , is less than

$$\begin{aligned} & \sum_{k=1}^{n_t} \mathbb{P} \left( R_k^t \leq (1+e^{-\tilde{c}h_t/2})/x_t, e_k S_k^t/t \geq x_t/4 \right) \mathbb{P} \left( H(\tilde{L}_{k-1})/t \leq 1 \right) + e^{-c\phi(t)} + P(V \notin \mathcal{G}_t) \\ & \leq \sum_{k=1}^{n_t} \mathbb{P} \left( R_k^t \leq (1+e^{-\tilde{c}h_t/2})/x_t \right) \mathbb{P}(e_k S_k^t/t \geq x_t/4) \mathbb{P} \left( H(\tilde{L}_{k-1})/t \leq 1 \right) + e^{-c\phi(t)} \end{aligned}$$

where we used the independence between  $R_k^t$  and  $e_k S_k^t$  and Lemma 2.6, and where the constant  $c$  has been suitably decreased. Since the sequence  $(e_k, S_k^t, R_k^t)_{k \geq 1}$  is *iid*, we deduce that the first part of the right-hand-side of (2.57),  $E[\sum_{k=1}^{n_t} \int_0^{1-4/\log(h_t)} P^V(\mathcal{E}_t(V, k, z), H(\tilde{m}_k)/t \in dz)]$ , is less than

$$\mathbb{P} \left( R_1^t \leq \frac{1+e^{-\tilde{c}h_t/2}}{x_t} \right) \times \mathbb{P}(e_1 S_1^t/t \geq x_t/4) \sum_{k=1}^{n_t} \mathbb{P} \left( H(\tilde{L}_{k-1})/t \leq 1 \right) + e^{-c\phi(t)}.$$

Using (2.37) and Lemma 2.12 to bound respectively the second and third factor of the second term, we get the existence of a positive constant  $C$  such that for  $t$  large enough,

$$E \left[ \sum_{k=1}^{n_t} \int_0^{1-4/\log(h_t)} P^V(\mathcal{E}_t(V, k, z), H(\tilde{m}_k)/t \in dz) \right] \leq \frac{C}{x_t^\kappa} \mathbb{P} \left( R_1^t \leq \frac{1 + e^{-\tilde{c}h_t/2}}{x_t} \right) + e^{-c\phi(t)} \quad (2.58)$$

It remains to study  $E[\sum_{k=1}^{n_t} \int_{1-4/\log(h_t)}^1 P^V(\mathcal{E}_t(V, k, z), H(\tilde{m}_k)/t \in dz)]$ . For  $v \in \mathcal{G}_t$  and  $z \in [1 - 4/\log(h_t), 1]$  we have, using the definitions of  $\mathcal{E}_t(v, k, z)$  and  $\mathcal{E}_t^8(v, k, z)$  :

$$\begin{aligned} \mathcal{E}_t(v, k, z) &\subset \left\{ \sup_{y \in \mathcal{D}_k} \mathcal{L}_{X_{\tilde{m}_k}}(4/\log(h_t), y) \geq tx_t, H_{X_{\tilde{m}_k}}(\tilde{L}_k) < H_{X_{\tilde{m}_k}}(\tilde{L}_{k-1}) \right\} \\ &\subset \left\{ \frac{4}{\log(h_t)} \frac{(1 + e^{-\tilde{c}h_t})}{(1 - e^{-\tilde{c}h_t})} \geq x_t R_k^t \right\} \cup \mathcal{E}_t^8(v, k, 1 - 4/\log(h_t)) \\ &\subset \{8/x_t \log(h_t) \geq R_k^t\} \cup \mathcal{E}_t^8(v, k, 1 - 4/\log(h_t)), \end{aligned}$$

where, in the last inclusion, we used the fact that  $(1 + e^{-\tilde{c}h_t})/(1 - e^{-\tilde{c}h_t}) \leq 2$  for large  $t$ . Recall that the sum in (2.57) corresponds to disjoint events, we thus get that  $\sum_{k=1}^{n_t} \int_{1-4/\log(h_t)}^1 P^v(\mathcal{E}_t(v, k, z), H(\tilde{m}_k)/t \in dz)$  is less than

$$\begin{aligned} &\sum_{k=1}^{n_t} P^v(R_k^t \leq 8/x_t \log(h_t), H(\tilde{m}_k)/t \in dz) + P^v(\cup_{k=1}^{n_t} \mathcal{E}_t^8(v, k, 1 - 4/\log(h_t))) \\ &\leq \sum_{k=1}^{n_t} \mathbb{1}_{R_k^t \leq 8/x_t \log(h_t)} \times P^v(H(\tilde{m}_k)/t \leq 1) + P^v(\cup_{k=1}^{n_t} \mathcal{E}_t^8(v, k, 1 - 4/\log(h_t))) \\ &\leq \sum_{k=1}^{n_t} \mathbb{1}_{R_k^t \leq 8/x_t \log(h_t)} \times P^v(H(\tilde{L}_{k-1})/t \leq 1) + e^{-c\phi(t)}, \end{aligned}$$

where, for the first inequality, we used the fact that  $R_k^t$  only depends on  $v$ , and for the second we used the fact that  $\tilde{L}_{k-1} \leq \tilde{m}_k$  for the first term and (2.35) for the second term. Here again, the first factor in the above product only depends on  $(\tilde{v}^{(k)}(x), \tilde{L}_{k-1} \leq x \leq \tilde{L}_k)$  (that is, only on  $v$  shifted at time  $\tilde{L}_{k-1}$ ) while the second factor depends on  $v$  before time  $\tilde{L}_{k-1}$ . As a consequence, when we integrate the above inequality with respect to  $v$  over  $D(\mathbb{R}, \mathbb{R}) = \mathcal{G}_t \cup \overline{\mathcal{G}_t}$  equipped with the probability measure  $P$ , we get that the two factors are independent so  $E[\sum_{k=1}^{n_t} \int_{1-4/\log(h_t)}^1 P^V(\mathcal{E}_t(V, k, z), H(\tilde{m}_k)/t \in dz)]$ , is less than

$$\begin{aligned} &\sum_{k=1}^{n_t} \mathbb{P}(R_k^t \leq 8/x_t \log(h_t)) \mathbb{P}(H(\tilde{L}_{k-1})/t \leq 1) + e^{-c\phi(t)} + P(V \notin \mathcal{G}_t) \\ &\leq \mathbb{P}(R_1^t \leq 8/x_t \log(h_t)) \sum_{k=1}^{n_t} \mathbb{P}(H(\tilde{L}_{k-1})/t \leq 1) + e^{-c\phi(t)}, \end{aligned}$$

where we used the fact that the sequence  $(R_k^t)_{k \geq 1}$  is *iid* and Lemma 2.6, and where the constant  $c$  has been suitably decreased. Note that from the definitions of  $x_t$  and  $h_t$  we have  $x_t \log(h_t) \sim K(\log(\log(t)))^\mu$ . Therefore, using Lemmas 2.13 (with  $z_t = x_t \log(h_t)/8$ ,  $a = 2$ ) and 2.12 to bound respectively the first and second factor we get that for  $t$  large enough the above is less than

$$C e^{\kappa\phi(t)} \mathbb{P}(I(V^\uparrow) \leq 1/K'(\log(\log(t)))^\mu) + C e^{\kappa\phi(t) - \delta\kappa h_t/3} + e^{-c\phi(t)},$$

for some positive constants  $C$  and  $K'$ . The term  $C e^{\kappa\phi(t) - \delta\kappa h_t/3}$  appears when  $V = W_\kappa$  (because of (2.50)) and it is not necessary otherwise, note that this term is ultimately less than  $e^{-c\phi(t)}$ .

Combining with (1.7) we get

$$E \left[ \sum_{k=1}^{n_t} \int_{1-4/\log(h_t)}^1 P^V(\mathcal{E}_t(V, k, z), H(\tilde{m}_k)/t \in dz) \right] \leq C e^{\kappa\phi(t) - K''(\log(\log(t)))^\mu} + 2e^{-c\phi(t)},$$

where  $K''$  is a positive constant. Since we have chosen  $\omega \in ]1, \mu[$  in (2.22) we have  $(\log(\log(t)))^\mu \gg \phi(t)$  so for  $t$  large enough the above inequality yields

$$E \left[ \sum_{k=1}^{n_t} \int_{1-4/\log(h_t)}^1 P^V(\mathcal{E}_t(V, k, z), H(\tilde{m}_k)/t \in dz) \right] \leq e^{-c\phi(t)}, \quad (2.59)$$

where the constant  $c$  has been suitably decreased. Now putting (2.58) and (2.59) in (2.57) we get for some constant  $c$  and  $t$  large enough :

$$\mathbb{P} \left( \sup_{\mathcal{D}_{N_t}} (\mathcal{L}_X(t, \cdot) - \mathcal{L}_X(H(\tilde{m}_{N_t}), \cdot)) \geq tx_t \right) \leq \mathbb{P} \left( R_1^t \leq \frac{u}{x_t} \right) + \mathbb{P}(N_t \geq n_t) + \mathbb{P}(\overline{\mathcal{E}_t^1}) + e^{-c\phi(t)}.$$

Using (2.23) and (2.25) we get the result for a suitably chosen constant  $c$  and  $t$  large enough.  $\square$

*Proof.* of Proposition 2.11

*Upper bound*

$$\begin{aligned} \mathbb{P}(\mathcal{L}_X^*(t) \geq tx_t) &\leq \mathbb{P}(\mathcal{L}_X^*(t) \geq tx_t, V \in \mathcal{V}_t, N_t < n_t, \mathcal{E}_t^1, \mathcal{E}_t^2) \\ &\quad + \mathbb{P}(V \notin \mathcal{V}_t) + \mathbb{P}(N_t \geq n_t) + \mathbb{P}(\overline{\mathcal{E}_t^1}) + \mathbb{P}(\overline{\mathcal{E}_t^2}). \end{aligned}$$

The event  $\mathcal{E}_t^1$  ensures that for  $j \leq n_t$ ,  $\tilde{L}_{j-1}$  is no longer reached after  $H(\tilde{m}_j)$  and the event  $\mathcal{E}_t^2$  ensures that the local time does not grow too much between  $H(\tilde{L}_{j-1})$  and  $H(\tilde{m}_j)$ . The event  $\{V \in \mathcal{V}_t, N_t < n_t\}$  ensures that at time  $t$  the diffusion is trapped in one of the first  $n_t$  standard valleys. The first term of the right hand side is thus less than

$$\begin{aligned} \mathbb{P} \left( \sup_{[\tilde{L}_{N_t-1}, \tilde{L}_{N_t}]} (\mathcal{L}_X(t, \cdot) - \mathcal{L}_X(H(\tilde{m}_{N_t}), \cdot)) \vee \sup_{1 \leq j \leq N_t-1} \sup_{[\tilde{L}_{j-1}, \tilde{L}_j]} (\mathcal{L}_X(H(\tilde{L}_j), \cdot) - \mathcal{L}_X(H(\tilde{m}_j), \cdot)) \geq t\tilde{x}_t^1, \right. \\ \left. V \in \mathcal{V}_t, N_t < n_t \right), \end{aligned}$$

where  $\tilde{x}_t^1 := x_t - e^{(\kappa(1+3\delta)-1)\phi(t)} \sim x_t$ . Then, since  $\tilde{x}_t^1$  converge to  $+\infty$ , we have  $\tilde{x}_t^1 \geq e^{-2\phi(t)}$  for  $t$  large enough. Using the definition of  $\mathcal{E}_t^3$ , we get that for such large  $t$  the above is less than

$$\begin{aligned} \mathbb{P} \left( \sup_{\mathcal{D}_{N_t}} (\mathcal{L}_X(t, \cdot) - \mathcal{L}_X(H(\tilde{m}_{N_t}), \cdot)) \vee \sup_{1 \leq j \leq N_t-1} \sup_{\mathcal{D}_j} (\mathcal{L}_X(H(\tilde{L}_j), \cdot) - \mathcal{L}_X(H(\tilde{m}_j), \cdot)) \geq t\tilde{x}_t^1, \right. \\ \left. V \in \mathcal{V}_t, N_t < n_t \right) + \mathbb{P}(\overline{\mathcal{E}_t^3}). \end{aligned}$$

Putting all this together, we see that for  $t$  large enough,  $\mathbb{P}(\mathcal{L}_X^*(t) \geq tx_t)$  is less than

$$\begin{aligned} \mathbb{P} \left( \sup_{1 \leq j \leq N_t-1} \sup_{\mathcal{D}_j} (\mathcal{L}_X(H(\tilde{L}_j), \cdot) - \mathcal{L}_X(H(\tilde{m}_j), \cdot)) \geq t\tilde{x}_t^1, V \in \mathcal{V}_t, N_t < n_t \right) \\ + \mathbb{P} \left( \sup_{\mathcal{D}_{N_t}} (\mathcal{L}_X(t, \cdot) - \mathcal{L}_X(H(\tilde{m}_{N_t}), \cdot)) \geq t\tilde{x}_t^1 \right) \\ + \mathbb{P}(V \notin \mathcal{V}_t) + \mathbb{P}(N_t \geq n_t) + \mathbb{P}(\overline{\mathcal{E}_t^1}) + \mathbb{P}(\overline{\mathcal{E}_t^2}) + \mathbb{P}(\overline{\mathcal{E}_t^3}). \end{aligned} \quad (2.60)$$



We now deal with the first term. Using the definition of  $\mathcal{E}_t^4$  we get that the first term of (2.60) is less than

$$\mathbb{P} \left( \sup_{1 \leq j \leq N_t-1} \mathcal{L}_X(H(\tilde{L}_j), \tilde{m}_j) \geq t\tilde{x}_t^2, V \in \mathcal{V}_t, N_t < n_t \right) + \mathbb{P}(\overline{\mathcal{E}_t^4}),$$

where  $\tilde{x}_t^2 := (1 + e^{-\tilde{c}ht})^{-1} \tilde{x}_t^1 \sim x_t$ . Now using the definition of  $\mathcal{E}_t^6$ , the above is less than

$$\mathbb{P} \left( \sup_{1 \leq j \leq N_t-1} e_j S_j^t \geq \tilde{x}_t, V \in \mathcal{V}_t, N_t < n_t \right) + \mathbb{P}(\overline{\mathcal{E}_t^4}) + \mathbb{P}(\overline{\mathcal{E}_t^6}),$$

where  $\tilde{x}_t := (1 + e^{-\tilde{c}ht})^{-1} \tilde{x}_t^2 \sim x_t$ . Using (2.26) with  $\eta = a - 1$ , the above is less than

$$\begin{aligned} & \mathbb{P} \left( \sup_{1 \leq j \leq N_{at}-1} e_j S_j^t \geq \tilde{x}_t \right) + \mathbb{P}(\overline{\mathcal{E}_t^4}) + \mathbb{P}(\overline{\mathcal{E}_t^6}) + \mathbb{P}(\overline{\mathcal{E}_t^5}) + \mathbb{P}(\overline{\mathcal{E}_t^7}), \\ &= \mathbb{P} \left( Y_1^{h,t} \left( Y_2^{-1,t}(a) - \right) \geq \tilde{x}_t \right) + \mathbb{P}(\overline{\mathcal{E}_t^4}) + \mathbb{P}(\overline{\mathcal{E}_t^6}) + \mathbb{P}(\overline{\mathcal{E}_t^5}) + \mathbb{P}(\overline{\mathcal{E}_t^7}). \end{aligned}$$

Combining with (2.60) and the fact that  $\tilde{x}_t^1 \geq \tilde{x}_t$ , we get that  $\mathbb{P}(\mathcal{L}_X^*(t) \geq tx_t)$  is less than

$$\begin{aligned} & \mathbb{P} \left( Y_1^{h,t} \left( Y_2^{-1,t}(a) - \right) \geq \tilde{x}_t \right) + \mathbb{P} \left( \sup_{\mathcal{D}_{N_t}} (\mathcal{L}_X(t, \cdot) - \mathcal{L}_X(H(\tilde{m}_{N_t}), \cdot)) \geq t\tilde{x}_t \right) + \mathbb{P}(V \notin \mathcal{V}_t) \\ & + \mathbb{P}(N_t \geq n_t) + \mathbb{P}(\overline{\mathcal{E}_t^1}) + \mathbb{P}(\overline{\mathcal{E}_t^2}) + \mathbb{P}(\overline{\mathcal{E}_t^3}) + \mathbb{P}(\overline{\mathcal{E}_t^4}) + \mathbb{P}(\overline{\mathcal{E}_t^5}) + \mathbb{P}(\overline{\mathcal{E}_t^6}) + \mathbb{P}(\overline{\mathcal{E}_t^7}). \end{aligned}$$

Applying Lemma 2.14 with  $u = \sqrt{a}$  (and  $x_t$  replaced by  $\tilde{x}_t$  which does not change anything since  $\tilde{x}_t$  also satisfies (2.27)), Fact 2.3, (2.23) and (2.25), we deduce the existence of a positive constant  $c$  such that for  $t$  large enough,

$$\mathbb{P}(\mathcal{L}_X^*(t) \geq tx_t) \leq \mathbb{P} \left( Y_1^{h,t} \left( Y_2^{-1,t}(a) - \right) \geq \tilde{x}_t \right) + \mathbb{P} \left( R_1^t \leq \frac{\sqrt{a}}{\tilde{x}_t} \right) + e^{-c\phi(t)}.$$

Since  $\tilde{x}_t \sim x_t$  and  $a > 1$  we get the upper bound when  $t$  is large enough.

*Lower bound*

$$\begin{aligned} \mathbb{P}(\mathcal{L}_X^*(t) \geq tx_t) & \geq \mathbb{P} \left( \sup_{1 \leq j \leq N_t-1} \mathcal{L}_X(t, \tilde{m}_j) \geq tx_t \right) \\ & \geq \mathbb{P} \left( \sup_{1 \leq j \leq N_t-1} \mathcal{L}_X(H(\tilde{L}_j), \tilde{m}_j) \geq tx_t, V \in \mathcal{V}_t, N_t < n_t, \mathcal{E}_t^1 \right) \end{aligned}$$

because, on  $\{V \in \mathcal{V}_t\} \cap \{N_t < n_t\} \cap \mathcal{E}_t^1$ ,  $\tilde{m}_j$  (for  $j < N_t$ ) is no longer reached between times  $H(\tilde{L}_j)$  and  $t$ ,

$$\geq \mathbb{P} \left( \sup_{1 \leq j \leq N_t-1} e_j S_j^t \geq t\hat{x}_t, V \in \mathcal{V}_t, N_t < n_t, \mathcal{E}_t^1, \mathcal{E}_t^6 \right)$$

because of the definition of  $\mathcal{E}_t^6$  and where  $\hat{x}_t := (1 - e^{-\tilde{c}ht})^{-1} x_t \sim x_t$ ,

$$\geq \mathbb{P} \left( \sup_{1 \leq j \leq N_{t/a}-1} e_j S_j^t \geq t\hat{x}_t, V \in \mathcal{V}_t, N_t < n_t, \mathcal{E}_t^1, \mathcal{E}_t^6, \mathcal{E}_t^5, \mathcal{E}_t^7 \right)$$

where we used (2.26) with  $\eta = 1 - a^{-1}$ ,

$$\geq \mathbb{P} \left( Y_1^{h,t} \left( Y_2^{-1,t}(1/a) - \right) \geq \hat{x}_t \right) - \left( \mathbb{P}(V \notin \mathcal{V}_t) + \mathbb{P}(N_t \geq n_t) + \mathbb{P}(\overline{\mathcal{E}_t^1}) + \mathbb{P}(\overline{\mathcal{E}_t^5}) + \mathbb{P}(\overline{\mathcal{E}_t^6}) + \mathbb{P}(\overline{\mathcal{E}_t^7}) \right).$$

Applying Fact 2.3, (2.23) and (2.25) we get the lower bound since  $\hat{x}_t \sim x_t$  and  $a > 1$ .

□

**Proposition 2.15.** *Let  $y_t$  go to infinity with  $t$ . For any  $b > 0$  and  $u \in ]0, 1[$  there is a positive constant  $C$  (depending on  $b$  and  $u$ ) such that for all  $t$  large enough,*

$$C \mathbb{P}(R_1^t \leq ub/y_t) / y_t^\kappa \leq \mathbb{P}\left(Y_1^{\natural,t}\left(Y_2^{-1,t}(b)-\right) \geq y_t\right) \leq \mathbb{P}(R_1^t \leq b/y_t).$$

*Proof.* For any  $z > 0$ , let  $k^t(z)$  be the first index  $k \geq 1$  such that  $e_k S_k^t / t \geq z$ . We have

$$\left\{Y_1^{\natural,t}\left(Y_2^{-1,t}(b)-\right) \geq y_t\right\} = \left\{k^t(y_t) < \mathcal{N}_{tb}\right\} = \left\{\sum_{i=1}^{k^t(y_t)} e_i S_i^t R_i^t \leq tb\right\} \subset \left\{e_{k^t(y_t)} S_{k^t(y_t)}^t R_{k^t(y_t)}^t \leq tb\right\} \quad (2.61)$$

and  $e_{k^t(y_t)} S_{k^t(y_t)}^t R_{k^t(y_t)}^t$  has the same law as  $e_1 S_1^t R_1^t$  conditionally on  $e_1 S_1^t \geq ty_t$  so, using (2.61) and the independence between  $e_1 S_1^t$  and  $R_1^t$  :

$$\mathbb{P}\left(Y_1^{\natural,t}\left(Y_2^{-1,t}(b)-\right) \geq y_t\right) \leq \mathbb{P}(e_1 S_1^t R_1^t \leq tb | e_1 S_1^t \geq ty_t) \leq \mathbb{P}(R_1^t \leq b/y_t),$$

which proves the upper bound. For the lower bound, we fix  $\eta < 1 - u$ . Note that according to (2.61),

$$\left\{e_{k^t(y_t)} S_{k^t(y_t)}^t R_{k^t(y_t)}^t \leq tb(1 - \eta)\right\} \cap \left\{\sum_{i=1}^{k^t(y_t)-1} e_i S_i^t R_i^t < tb\eta\right\} \subset \left\{Y_1^{\natural,t}\left(Y_2^{-1,t}(b)-\right) \geq y_t\right\},$$

and both events on the left-hand-side are independent so

$$\mathbb{P}\left(Y_1^{\natural,t}\left(Y_2^{-1,t}(b)-\right) \geq y_t\right) \geq \mathbb{P}(e_1 S_1^t R_1^t \leq tb(1 - \eta) | e_1 S_1^t \geq ty_t) \times \mathbb{P}\left(\sum_{i=1}^{k^t(y_t)-1} e_i S_i^t R_i^t < tb\eta\right) \quad (2.62)$$

We first deal with the second factor. Let  $(Z_i)_{i \geq 1}$  be *iid* random variables such that  $\mathcal{L}(Z_1) = \mathcal{L}(e_1 S_1^t R_1^t | e_1 S_1^t \geq ty_t)$  and  $T$  be a geometric random variable with parameter  $\mathbb{P}(e_1 S_1^t \geq ty_t)$ , independent from the sequences  $(Z_i)_{i \geq 1}$  and  $(e_i S_i^t R_i^t)_{i \geq 1}$ . We then have

$$\mathbb{P}\left(\sum_{i=1}^{k^t(y_t)-1} e_i S_i^t R_i^t < tb\eta\right) = \mathbb{P}\left(\sum_{i=1}^{T-1} Z_i < tb\eta\right) \geq \mathbb{P}\left(\sum_{i=1}^{T-1} e_i S_i^t R_i^t < tb\eta\right),$$

because the random variable  $e_i S_i^t R_i^t$  is stochastically greater than the random variable  $Z_i$ . Then,

$$\mathbb{P}\left(\sum_{i=1}^{T-1} e_i S_i^t R_i^t < tb\eta\right) \geq \mathbb{P}(T \leq e^{\kappa\phi(t)}) \times \mathbb{P}\left(\sum_{1 \leq i \leq e^{\kappa\phi(t)}} e_i S_i^t R_i^t < tb\eta\right).$$

On the first hand we have

$$\mathbb{P}\left(\sum_{1 \leq i \leq e^{\kappa\phi(t)}} e_i S_i^t R_i^t < tb\eta\right) = \mathbb{P}(Y_2^t(1) < b\eta) \xrightarrow{t \rightarrow +\infty} \mathbb{P}(\mathcal{Y}_2(1) < b\eta) > 0,$$

where we used Fact 2.4 and where  $\mathcal{Y}_2$  is as in there, the second component of the limit process  $(\mathcal{Y}_1, \mathcal{Y}_2)$ . On the second hand

$$\mathbb{P}(T \leq e^{\kappa\phi(t)}) = 1 - (1 - \mathbb{P}(e_1 S_1^t \geq ty_t))^{[e^{\kappa\phi(t)}]} = 1 - e^{[e^{\kappa\phi(t)}] \ln(1 - \mathbb{P}(e_1 S_1^t \geq ty_t))}.$$

Using (2.37) we get  $\mathbb{P}(T \leq e^{\kappa\phi(t)}) \underset{t \rightarrow +\infty}{\sim} \mathcal{C}'/y_t^\kappa$ . Putting all this together, we get the existence of a positive constant  $c_1 > 0$  such that for  $t$  large enough,

$$\mathbb{P}\left(\sum_{i=1}^{k^t(y_t)-1} e_i S_i^t R_i^t < tb\eta\right) \geq c_1/y_t^\kappa. \quad (2.63)$$

We now study the first factor in the right hand side of (2.62). From the independence of the two factors  $e_1 S_1^t$  and  $R_1^t$  in  $e_1 S_1^t R_1^t$  we have

$$\mathbb{P}(e_1 S_1^t R_1^t \leq tb(1-\eta) | e_1 S_1^t \geq ty_t) \geq \mathbb{P}(R_1^t \leq bu/y_t) \times \mathbb{P}(e_1 S_1^t \leq ty_t(1-\eta)/u | e_1 S_1^t \geq ty_t)$$

and

$$\begin{aligned} \mathbb{P}(e_1 S_1^t \leq ty_t(1-\eta)/u | e_1 S_1^t \geq ty_t) &= \frac{\mathbb{P}(ty_t \leq e_1 S_1^t \leq ty_t(1-\eta)/u)}{\mathbb{P}(e_1 S_1^t \geq ty_t)} \\ &= 1 - \frac{\mathbb{P}(e_1 S_1^t > ty_t(1-\eta)/u)}{\mathbb{P}(e_1 S_1^t \geq ty_t)} \\ &\xrightarrow{t \rightarrow +\infty} 1 - (u/(1-\eta))^\kappa > 0, \end{aligned}$$

where the limit comes from (2.37), because  $y_t$  goes to infinity. We thus get the existence of a positive constant  $c_2 > 0$  such that for  $t$  large enough,

$$\mathbb{P}(e_1 S_1^t R_1^t \leq tb(1-\eta) | e_1 S_1^t \geq ty_t) \geq c_2 \mathbb{P}(R_1^t \leq bu/y_t). \quad (2.64)$$

Putting (2.63) and (2.64) in (2.62) we get the lower bound.

□

Fix  $\theta > 1$ . We apply Proposition 2.11 (the upper bound with  $a = \theta^{1/3}$  and the lower bound with  $a = \theta^{1/4}$ ), Proposition 2.15 (the upper bound applied with  $b = \theta^{1/3}$ ,  $y_t = \theta^{-1/3}x_t$  and the lower bound with  $b = \theta^{-1/4}$ ,  $y_t = \theta^{1/4}x_t$ ,  $u = \theta^{-1/4}$ ) and Lemma 2.13 (the upper bounds of (2.49) and (2.50) applied with  $a = \theta^{1/3}$ ,  $z_t = \theta^{-2/3}x_t$ , and the lower bounds of these same expressions applied with  $a = \theta^{1/4}$ ,  $z_t = \theta^{3/4}x_t$ ). We get :

**Proposition 2.16.** *Fix  $\theta > 1$ . If  $V$  possesses negative jumps, there are positive constants  $\tilde{C}$  and  $c$  such that for  $t$  large enough,*

$$\frac{e^{-c(\log(x_t))^2}}{\tilde{C}x_t^\kappa} \mathbb{P}\left(I(V^\uparrow) \leq 1/\theta x_t\right) - e^{-c\phi(t)} \leq \mathbb{P}(\mathcal{L}_X^*(t) \geq tx_t) \leq \tilde{C} \mathbb{P}\left(I(V^\uparrow) \leq \theta/x_t\right) + e^{-c\phi(t)} \quad (2.65)$$

If  $V := W_\kappa$ , the  $\kappa$ -drifted brownian motion, there are positive constants  $C$  and  $c$  such that for  $t$  large enough,

$$\frac{1}{\tilde{C}x_t^\kappa} \mathbb{P}(\mathcal{R} \leq 1/\theta x_t) - e^{-c\phi(t)} \leq \mathbb{P}(\mathcal{L}_X^*(t) \geq tx_t) \leq \tilde{C} \mathbb{P}(\mathcal{R} \leq \theta/x_t) + e^{-c\phi(t)}. \quad (2.66)$$

We can now link the asymptotic behavior of the local time with the left tail of  $I(V^\uparrow)$  :

*Proof.* of Theorem 1.2

First, we assume that  $V$  possesses negative jumps.

Let us assume that (1.12) is satisfied with some constants  $\gamma > 1$  and  $C > 0$ . We now prove (1.13). Let  $a > 1$  and define the events

$$\mathcal{A}_n := \left\{ \sup_{t \in [a^n, a^{n+1}]} \frac{\mathcal{L}_X^*(t)}{t(\log(\log(t)))^{\gamma-1}} \geq a^3 C^{1-\gamma} \right\}.$$

We define  $x_t := C^{1-\gamma} a^2 (\log(\log(t/a)))^{\gamma-1}$ . Note that such a choice of  $x_t$  satisfies (2.27) with  $\mu = \gamma$  and  $D = C^{1-\gamma}$ . From the increase of  $\mathcal{L}_X^*(\cdot)$ , (2.65) (the upper bound applied with  $t = a^{n+1}$ ,  $\theta = a$ ) and (1.12) we have, for  $n$  large enough,

$$\begin{aligned} \mathbb{P}(\mathcal{A}_n) &\leq \mathbb{P}(\mathcal{L}_X^*(a^{n+1}) \geq C^{1-\gamma} a^{n+3} (\log(\log(a^n)))^{\gamma-1}) \\ &\leq \tilde{C} \mathbb{P}(I(V^\uparrow) \leq 1/a C^{1-\gamma} (\log(\log(a^n)))^{\gamma-1}) + e^{-c\phi(a^{n+1})} \\ &\leq \tilde{C} \exp\left(-a^{\frac{1}{\gamma-1}} \log(\log(a^n))\right) + e^{-c\phi(a^{n+1})} \\ &= \tilde{C} (\log(a))^{-a^{\frac{1}{\gamma-1}}} n^{-a^{\frac{1}{\gamma-1}}} + e^{-c\phi(a^{n+1})}. \end{aligned}$$

Since  $e^{-c\phi(a^{n+1})} = e^{-c(\log \log(a^{n+1}))^\omega} \leq n^{-2}$  for  $n$  large enough, the above is the general term of a converging series so, using the Borel-Cantelli lemma, we deduce that  $\mathbb{P}$ -almost surely,

$$\limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t(\log(\log(t)))^{\gamma-1}} \leq a^3 C^{1-\gamma},$$

and letting  $a$  go to 1 we get (1.13).

Now, let us assume that (1.14) is satisfied with some constants  $\gamma > 1$  and  $C > 0$ . We now prove (1.15). Let  $a > 1$  and let  $t_n$ ,  $u_n$ ,  $v_n$  (defined from this  $a$ ) and  $X^n$  be as in Subsection 2.2 (recall that  $X^n - v_n$  is equal in law to  $X$  under the annealed probability  $\mathbb{P}$ ). We define

$$\mathcal{B}_n := \left\{ \frac{\mathcal{L}_{X^n}^*(t_n)}{t_n(\log(\log(t_n)))^{\gamma-1}} \geq \frac{C^{1-\gamma}}{a^{3(\gamma-1)}} \right\}.$$

We also define  $\mathcal{C}_n$  and  $\mathcal{D}_n$  to be as in Lemma 2.10 and  $\mathcal{E}_n := \mathcal{B}_n \cap \mathcal{C}_n$ . We define  $x_t := C^{1-\gamma} (\log(\log(t)))^{\gamma-1} / a^{3(\gamma-1)}$ . Note that such a choice of  $x_t$  satisfies (2.27) with  $\mu = \gamma$  and  $D = C^{1-\gamma} / a^{3(\gamma-1)}$ . According to (2.65) (the lower bound applied with  $t = t_n$ ,  $\theta = a^{\gamma-1}$ ), the definition of  $t_n$ , (1.14) and the fact that  $n$  is large we have, for some positive constants  $c_a$ ,  $K_a$  and  $n$  large enough,

$$\begin{aligned} \mathbb{P}(\mathcal{B}_n) &\geq K_a e^{-c_a(\log(\log(\log(t_n))))^2} \mathbb{P}\left(I(V^\uparrow) \leq a^{2(\gamma-1)} / C^{1-\gamma} (\log(\log(t_n)))^{\gamma-1}\right) / (\log(\log(t_n)))^{\kappa(\gamma-1)} - e^{-c\phi(t_n)} \\ &= K_a e^{-c_a(\log(a \log(n)))^2} \mathbb{P}\left(I(V^\uparrow) \leq a^{\gamma-1} / C^{1-\gamma} (\log(n))^{\gamma-1}\right) / (a \log(n))^{\kappa(\gamma-1)} - e^{-c\phi(t_n)} \\ &\geq K_a e^{-c_a(\log(a \log(n)))^2} e^{-(\log(n))/a} / (a \log(n))^{\kappa(\gamma-1)} - e^{-c\phi(t_n)} \\ &= K_a \exp\left(-c_a(\log(a \log(n)))^2 - \log(n)/a + \kappa(\gamma-1) \log(a \log(n))\right) - e^{-c\phi(t_n)} \\ &\geq K_a \exp(-\log(n)) - e^{-c\phi(t_n)} \\ &= K_a n^{-1} - e^{-c\phi(t_n)}. \end{aligned}$$

Since  $e^{-c\phi(t_n)} = e^{-c(\log \log(t_n))^\omega} \leq n^{-2}$  for  $n$  large enough, we get

$$\sum_{n \geq 1} \mathbb{P}(\mathcal{B}_n) = +\infty. \quad (2.67)$$

Then, the combination of (2.67) and (2.44) yields

$$\sum_{n \geq 1} \mathbb{P}(\mathcal{E}_n) \geq \sum_{n \geq 1} \mathbb{P}(\mathcal{B}_n) - \sum_{n \geq 1} \mathbb{P}(\overline{\mathcal{C}_n}) = +\infty. \quad (2.68)$$

Note that each event  $\mathcal{E}_n$  belongs to the  $\sigma$ -field  $\sigma(V(s) - V(u_n), u_n \leq s \leq u_{n+1}, X(t), H(v_n) \leq t \leq H(v_n) + T_n)$ , in other words, it only depends on the diffusion between times  $H(v_n)$  and  $H(v_n) + T_n$  and on the environment between positions  $u_n$  and  $u_{n+1}$ . From the Markov property and the independence of the increments of the environment, we get that the events  $(\mathcal{E}_n)_{n \geq 1}$  are independent. Combining this independence with (2.68) and the Borel-Cantelli Lemma we get that  $\mathbb{P}$ -almost surely, the event  $\mathcal{E}_n$  is realized infinitely many often. For  $n$  such that this event is realized we have

$$\frac{\mathcal{L}_X^*(H(v_n) + t_n)}{t_n(\log(\log(t_n)))^{\gamma-1}} \geq \frac{\mathcal{L}_{X_n}^*(t_n)}{t_n(\log(\log(t_n)))^{\gamma-1}} \geq \frac{C^{1-\gamma}}{a^{3(\gamma-1)}}. \quad (2.69)$$

According to (2.45) and the Borel-Cantelli Lemma we have  $\mathbb{P}$ -almost surely

$$H(v_n) + t_n \underset{n \rightarrow +\infty}{\sim} t_n,$$

so combining with (2.69) we deduce that  $\mathbb{P}$ -almost surely,

$$\limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t(\log(\log(t)))^{\gamma-1}} \geq \frac{C^{1-\gamma}}{a^{3(\gamma-1)}},$$

and letting  $a$  go to 1 we get (1.15).

If  $V = W_\kappa$ , the  $\kappa$ -drifted brownian motion with  $0 < \kappa < 1$ , we proceed the same proof, only replacing  $I(V^\uparrow)$  by  $\mathcal{R}$  and (2.65) by (2.66). We thus get that the same result is still true for  $V = W_\kappa$ , but with  $\mathcal{R}$  instead of  $I(V^\uparrow)$ . □

The other theorems for the lim sup are now easy to prove.

*Proof.* of Theorems 1.4 and 1.5

In the case where  $V$  possesses negative jumps, Theorem 1.4 is a direct consequence of the combination of Theorem 1.2, (1.8) and (1.9). Similarly, the first point of Theorem 1.5 is obtained from the combination of Theorem 1.2 and (1.10). The second point of Theorem 1.5 is obtained from the combination of Theorem 1.2 and (1.11).

In the case where  $V = W_\kappa$ , the  $\kappa$ -drifted brownian motion with  $0 < \kappa < 1$ , we only need to prove the last point of Theorem 1.5, and this requires to determine exactly the left tail of  $\mathcal{R}$ . This variable is equal in law to the sum of two independent random variables having the same law as  $I(W_k^\uparrow)$ . We thus have

$$-\log(\mathbb{E}[e^{-\lambda \mathcal{R}}]) = -\log\left(\left(\mathbb{E}[e^{-\lambda I(W_k^\uparrow)}]\right)^2\right) = -2\log\left(\mathbb{E}[e^{-\lambda I(W_k^\uparrow)}]\right) \underset{\lambda \rightarrow +\infty}{\sim} 4\sqrt{2\lambda},$$

where the equivalent comes from (1.13) of [13]. Using this together with De Bruijn's Theorem (see Theorem 4.12.9 in [5]) we get

$$-\log(\mathbb{P}(\mathcal{R} \leq x)) \underset{x \rightarrow 0}{\sim} \frac{8}{x}.$$

The last point of Theorem 1.5 follows from this combined with Theorem 1.2. □

**2.4. The lim inf.** For the lim inf we study of the asymptotic of the quantity  $\mathbb{P}(\mathcal{L}_X^*(t)/t \leq 1/x_t)$ . Recall that  $x_t$  is defined in (2.27) where  $D > 0$  and  $\mu \in ]1, 2]$  are fixed constants. In all this subsection we take  $\omega := 2$  for the parameter in (2.22).

**Proposition 2.17.** *Recall the  $\lambda_0$  defined in Fact 2.8. There is a positive constant  $c$  such that for all  $a > 1$  and  $t$  large enough we have,*

$$\begin{aligned} \mathbb{P}\left(Y_1^{\natural,t}\left(Y_2^{-1,t}(a)\right) \leq 1/ax_t\right) - e^{-c\phi(t)} &\leq \mathbb{P}\left(\mathcal{L}_X^*(t) \leq t/x_t\right) \\ &\leq 2\mathbb{P}\left(Y_1^{\natural,t}\left(Y_2^{-1,t}(1/4)\right) \leq 2/x_t\right) + e^{-\lambda_0 x_t/8} + e^{-c\phi(t)}. \end{aligned}$$

Here, the functional of  $(Y_1^t, Y_2^t)$  involved is  $Y_1^{\natural,t}(Y_2^{-1,t}(\cdot))$  which represents the supremum of the local time after leaving the last valley.

*Proof. Lower bound*

From the definition of  $N_t$ , we have  $H(\tilde{m}_{N_t+1}) \geq t$  on  $\{V \in \mathcal{V}_t, N_t < n_t\}$  so

$$\mathbb{P}\left(\mathcal{L}_X^*(t) \leq t/x_t\right) \geq \mathbb{P}\left(\mathcal{L}_X^*(H(\tilde{m}_{N_t+1})) \leq t/x_t, V \in \mathcal{V}_t, N_t < n_t, \mathcal{E}_t^1, \mathcal{E}_t^2\right).$$

The event  $\mathcal{E}_t^1$  ensures that for  $j \leq n_t$ ,  $\tilde{L}_{j-1}$  is no longer reached after  $H(\tilde{m}_j)$  and the event  $\mathcal{E}_t^2$  ensures that the local time does not grow too much between  $H(\tilde{L}_{j-1})$  and  $H(\tilde{m}_j)$ . The event  $\{V \in \mathcal{V}_t, N_t < n_t\}$  ensures that at time  $t$  the diffusion is trapped in one of the first  $n_t$  standard valleys. The right hand side is thus more than

$$\mathbb{P}\left(\sup_{1 \leq j \leq N_t} \sup_{[\tilde{L}_{j-1}, \tilde{L}_j]} \left(\mathcal{L}_X(H(\tilde{L}_j), \cdot) - \mathcal{L}_X(H(\tilde{m}_j), \cdot)\right) \leq t/\tilde{x}_t^1, V \in \mathcal{V}_t, N_t < n_t, \mathcal{E}_t^1, \mathcal{E}_t^2\right),$$

where  $\tilde{x}_t^1 := 1/((1/x_t) - e^{(\kappa(1+3\delta)-1)\phi(t)})$ . Then, since  $\tilde{x}_t^1 \sim x_t$ , we have  $1/\tilde{x}_t^1 \geq e^{-2\phi(t)}$  for  $t$  large enough. Using the definition of  $\mathcal{E}_t^3$ , we get that for such large  $t$  the above is more than

$$\mathbb{P}\left(\sup_{1 \leq j \leq N_t} \sup_{\mathcal{D}_j} \left(\mathcal{L}_X(H(\tilde{L}_j), \cdot) - \mathcal{L}_X(H(\tilde{m}_j), \cdot)\right) \leq t/\tilde{x}_t^1, V \in \mathcal{V}_t, N_t < n_t, \mathcal{E}_t^1, \mathcal{E}_t^2, \mathcal{E}_t^3\right).$$

From the definition of  $\mathcal{E}_t^4$  the above is more than

$$\mathbb{P}\left(\sup_{1 \leq j \leq N_t} \mathcal{L}_X(H(\tilde{L}_j), \tilde{m}_j) \leq t/\tilde{x}_t^2, V \in \mathcal{V}_t, N_t < n_t, \mathcal{E}_t^1, \mathcal{E}_t^2, \mathcal{E}_t^3, \mathcal{E}_t^4\right),$$

where  $\tilde{x}_t^2 := (1 + e^{-\tilde{c}ht})\tilde{x}_t^1 \sim x_t$ . Now using the definition of  $\mathcal{E}_t^6$ , the above is more than

$$\mathbb{P}\left(\sup_{1 \leq j \leq N_t} e_j S_j^t \leq 1/\tilde{x}_t, V \in \mathcal{V}_t, N_t < n_t, \mathcal{E}_t^1, \mathcal{E}_t^2, \mathcal{E}_t^3, \mathcal{E}_t^4, \mathcal{E}_t^6\right),$$

where  $\tilde{x}_t := (1 + e^{-\tilde{c}ht})\tilde{x}_t^2 \sim x_t$ . Let  $a > 1$ . Using (2.26) with  $\eta = a - 1$ , the above is more than

$$\begin{aligned} &\mathbb{P}\left(\sup_{1 \leq j \leq N_{at}} e_j S_j^t \leq 1/\tilde{x}_t, V \in \mathcal{V}_t, N_t < n_t, \mathcal{E}_t^1, \mathcal{E}_t^2, \mathcal{E}_t^3, \mathcal{E}_t^4, \mathcal{E}_t^5, \mathcal{E}_t^6, \mathcal{E}_t^7\right) \\ &\geq \mathbb{P}\left(Y_1^{\natural,t}\left(Y_2^{-1,t}(a)\right) \leq 1/\tilde{x}_t\right) \\ &\quad - \left(\mathbb{P}(V \notin \mathcal{V}_t) + \mathbb{P}(N_t \geq n_t) + \mathbb{P}(\overline{\mathcal{E}_t^1}) + \mathbb{P}(\overline{\mathcal{E}_t^2}) + \mathbb{P}(\overline{\mathcal{E}_t^3}) + \mathbb{P}(\overline{\mathcal{E}_t^4}) + \mathbb{P}(\overline{\mathcal{E}_t^5}) + \mathbb{P}(\overline{\mathcal{E}_t^6}) + \mathbb{P}(\overline{\mathcal{E}_t^7})\right), \end{aligned}$$

where we used the definition of  $(Y_1^t, Y_2^t)$ . Applying Fact 2.3, (2.23) and (2.25) we get the asserted lower bound for a suitably chosen constant  $c$  and  $t$  large enough, since  $\tilde{x}_t \sim x_t$  and  $a > 1$ .

Upper bound

$$\begin{aligned} \mathbb{P}(\mathcal{L}_X^*(t) \leq t/x_t) &\leq \mathbb{P}\left(\sup_{1 \leq j \leq N_t} \mathcal{L}_X(t, \tilde{m}_j) \leq t/x_t, V \in \mathcal{V}_t, N_t < n_t\right) + \mathbb{P}(V \notin \mathcal{V}_t) + \mathbb{P}(N_t \geq n_t) \\ &\leq \mathbb{P}\left(\mathcal{L}_X(t, \tilde{m}_{N_t}) \leq t/x_t, \sup_{1 \leq j \leq N_t-1} \mathcal{L}_X(H(\tilde{L}_j), \tilde{m}_j) \leq t/x_t, V \in \mathcal{V}_t, N_t < n_t, \mathcal{E}_t^1\right) \\ &\quad + \mathbb{P}(\overline{\mathcal{E}_t^1}) + \mathbb{P}(V \notin \mathcal{V}_t) + \mathbb{P}(N_t \geq n_t), \end{aligned} \quad (2.70)$$

because, on  $\{V \in \mathcal{V}_t, N_t < n_t\} \cap \mathcal{E}_t^1$ ,  $\tilde{m}_j$  (for  $j < N_t$ ) is no longer reached between times  $H(\tilde{L}_j)$  and  $t$ . We fix  $v \in \mathcal{G}_t$ , a realization of the environment. Let us define

$$\mathcal{E}_t(v, k, z) := \left\{ \mathcal{L}_{X_{\tilde{m}_k}}(t(1-z), \tilde{m}_k) \leq t/x_t, H_{X_{\tilde{m}_k}}(\tilde{m}_{k+1}) \geq t(1-z), H_{X_{\tilde{m}_k}}(\tilde{L}_k) < H_{X_{\tilde{m}_k}}(\tilde{L}_{k-1}) \right\},$$

and  $\nu_t(v, k, z) := P^v(\mathcal{E}_t(v, k, z))$ . The event  $\mathcal{E}_t(v, k, z)$  belongs to the  $\sigma$ -field  $\sigma(X(t), t \geq H(\tilde{m}_k))$ . In other words, it only depends on the diffusion after time  $H(\tilde{m}_k)$ . On the other hand,  $H(\tilde{m}_k)$  is measurable with respect to the  $\sigma$ -field  $\sigma(X(t), 0 \leq t \leq H(\tilde{m}_k))$ . From the Markov property applied to  $X$  at  $H(\tilde{m}_k)$ , we get that  $H(\tilde{m}_k)$  is independent from the event  $\mathcal{E}_t(v, k, z)$ . As a consequence,  $P^v(\mathcal{L}_X(t, \tilde{m}_{N_t}) \leq t/x_t, \sup_{1 \leq j \leq N_t-1} \mathcal{L}_X(H(\tilde{L}_j), \tilde{m}_j) \leq t/x_t, N_t < n_t, \mathcal{E}_t^1)$  is less than

$$\sum_{k=1}^{n_t} \int_0^1 \nu_t(v, k, z) \times P^v \left( \sup_{1 \leq j \leq k-1} \mathcal{L}_X(H(\tilde{L}_j), \tilde{m}_j) \leq t/x_t, H(\tilde{m}_k)/t \in dz \right). \quad (2.71)$$

The fact that the sum stops at  $n_t$  comes from  $N_t < n_t$  together with the fact that  $v \in \mathcal{G}_t \subset \mathcal{V}_t$ . From the definition of  $\mathcal{E}_t^9(v, k, z)$  we get

$$\mathcal{E}_t(v, k, z) \subset \left\{ \frac{1-z}{R_k^t} (1 - e^{-\tilde{c}h_t}) \leq \frac{1}{x_t}, H_{X_{\tilde{m}_k}}(\tilde{m}_{k+1}) \geq t(1-z) \right\} \cup \mathcal{E}_t^9(v, k, z)$$

and,  $v$  being fixed, the events in the above expression are independent from the  $\sigma$ -field  $\sigma(X(t), 0 \leq t \leq H(\tilde{m}_k))$ , so putting into (2.71), using the independence and the fact that the sum in (2.71) corresponds to disjoint events (so it is actually the probability of a union of events), we get that the first term in the right hand side of (2.70) is less than

$$\begin{aligned} &\mathbb{P} \left( \frac{1 - H(\tilde{m}_{N_t})/t}{R_{N_t}^t} (1 - e^{-\tilde{c}h_t}) \leq 1/x_t, \sup_{1 \leq j \leq N_t-1} \mathcal{L}_X(H(\tilde{L}_j), \tilde{m}_j) \leq t/x_t \right) \\ &+ E \left[ \mathbb{1}_{V \in \mathcal{G}_t} P^V \left( \cup_{k=1}^{n_t} \{N_t \geq k\} \cap \mathcal{E}_t^9(V, k, H(\tilde{m}_k)/t) \right) \right] + P(V \notin \mathcal{G}_t) \\ &\leq \mathbb{P} \left( (1 - e^{-\tilde{c}h_t})^{-1} \frac{R_{N_t}^t}{x_t} + \frac{H(\tilde{m}_{N_t})}{t} \geq 1, \sup_{1 \leq j \leq N_t-1} \mathcal{L}_X(H(\tilde{L}_j), \tilde{m}_j) \leq t/x_t \right) + e^{-c\phi(t)}, \end{aligned}$$

where  $c$  is a positive constant and where we used (2.36) for the second term, Lemma 2.6 for the third term and the fact that  $t$  is large enough,

$$\leq \mathbb{P} \left( (1 - e^{-\tilde{c}h_t})^{-1} \frac{R_{N_t}^t}{x_t} + \frac{(1 + e^{-\tilde{c}h_t})}{t} \sum_{j=1}^{N_t-1} e_j S_j^t R_j^t \geq 1 - 2/\log h_t, \sup_{1 \leq j \leq N_t-1} e_j S_j^t \leq (1 - e^{-\tilde{c}h_t})^{-1} t/x_t, \right.$$

$$\left. V \in \mathcal{V}_t, N_t < n_t, \mathcal{E}_t^5, \mathcal{E}_t^6, \mathcal{E}_t^7 \right) + \mathbb{P}(V \notin \mathcal{V}_t) + \mathbb{P}(N_t \geq n_t) + \mathbb{P}(\overline{\mathcal{E}_t^5}) + \mathbb{P}(\overline{\mathcal{E}_t^6}) + \mathbb{P}(\overline{\mathcal{E}_t^7}) + e^{-c\phi(t)},$$

where we used the definitions of  $\mathcal{E}_t^5$ ,  $\mathcal{E}_t^6$  and  $\mathcal{E}_t^7$ ,

$$\leq \mathbb{P} \left( \frac{R_{N_t}^t}{x_t} + \frac{1}{t} \sum_{j=1}^{N_t-1} e_j S_j^t R_j^t \geq 1/2, \sup_{1 \leq j < N_t-1} e_j S_j^t < 2t/x_t, V \in \mathcal{V}_t, N_t < n_t, \mathcal{E}_t^5, \mathcal{E}_t^6, \mathcal{E}_t^7 \right) + e^{-c\phi(t)}$$



where we used the fact that  $t$  is large enough, Fact 2.3, (2.23) and (2.25), and where the constant  $c$  has been suitably decreased,

$$\begin{aligned} &\leq \mathbb{P} \left( \frac{R_{N_t}^t}{x_t} + \frac{1}{t} \sum_{j=1}^{N_t-1} e_j S_j^t R_j^t \geq 1/2, \sup_{1 \leq j \leq N_t-1} e_j S_j^t < 2t/x_t, e_{N_t} S_{N_t}^t \geq 2t/x_t \right) \\ &+ \mathbb{P} \left( \sup_{1 \leq j \leq N_t} e_j S_j^t < 2t/x_t, V \in \mathcal{V}_t, N_t < n_t, \mathcal{E}_t^5, \mathcal{E}_t^6, \mathcal{E}_t^7 \right) + e^{-c\phi(t)}. \end{aligned}$$

On the event in the probability of the first term, we have  $N_t = k^t(2/x_t)$ . For the second term we use (2.26) with  $\eta = 3/4$ . The above is thus less than

$$\begin{aligned} &\mathbb{P} \left( \frac{R_{k^t(2/x_t)}^t}{x_t} + \frac{1}{t} \sum_{j=1}^{k^t(2/x_t)-1} e_j S_j^t R_j^t \geq 1/2 \right) + \mathbb{P} \left( \sup_{1 \leq j \leq N_{t/4}} e_j S_j^t < 2t/x_t \right) + e^{-c\phi(t)} \\ &\leq \mathbb{P} \left( \frac{R_{k^t(2/x_t)}^t}{x_t} \geq 1/4 \right) + \mathbb{P} \left( \frac{1}{t} \sum_{j=1}^{k^t(2/x_t)-1} e_j S_j^t R_j^t \geq 1/4 \right) + \mathbb{P} \left( \sup_{1 \leq j \leq N_{t/4}} e_j S_j^t < 2t/x_t \right) + e^{-c\phi(t)} \\ &\leq \mathbb{P} (R_1^t \geq x_t/4) + 2\mathbb{P} \left( Y_1^{t,t} \left( Y_2^{-1,t}(1/4) \right) \leq 2/x_t \right) + e^{-c\phi(t)} \end{aligned}$$

where, for the last inequality, we used the fact that the sequence  $(R_j^t)_{j \geq 1}$  is *iid* and independent from the random index  $k^t(2/x_t)$ , (2.81) with  $b = 1/4$ ,  $y_t = x_t/2$ , and the definition of  $(Y_1^t, Y_2^t)$ . Then, note that according to (2.39) we have for all  $t$  large enough,

$$\mathbb{P} (R_1^t \geq x_t/4) \leq e^{-\lambda_0 x_t/8}.$$

Bounding the three terms in the right hand side of (2.70) thanks to the above, Fact 2.3, (2.23) and (2.25) we get the upper bound for a suitably chosen constant  $c$  and  $t$  large enough.  $\square$

We now study the functional involved in Proposition 2.17. For this we need two lemmas.

In the remaining part of this subsection we fix  $\eta \in ]0, 1/3[$ . Let  $y_t$  go to infinity with  $t$  satisfying  $\log(y_t) \ll \phi(t)$ . Let  $p_t := \mathbb{P}(e_1 S_1^t > t/y_t)$  and  $(\overline{\mathcal{H}}_i)_{i \geq 1}$  be *iid* random variables such that  $\overline{\mathcal{H}}_1$  has the same law as  $e_1 S_1^t R_1^t$  conditionally to  $\{e_1 S_1^t \leq t/y_t\}$ :  $\mathcal{L}(\overline{\mathcal{H}}_1) = \mathcal{L}(e_1 S_1^t R_1^t | e_1 S_1^t \leq t/y_t)$ . Since we have  $\log(y_t) \ll \phi(t)$ , (2.37) gives  $p_t \sim \mathcal{C}' e^{-\kappa\phi(t)} y_t^\kappa$ . We have

**Lemma 2.18.**

$$1 - \mathbb{E} \left[ e^{-\lambda \overline{\mathcal{H}}_1/t} \right] \underset{t \rightarrow +\infty}{\sim} \lambda \frac{\mathcal{C}' \kappa \mathbb{E}[\mathcal{R}]}{(1 - \kappa) e^{\kappa\phi(t)} y_t^{1-\kappa}}. \quad (2.72)$$

*Proof.* For any  $\lambda \geq 0$  we have

$$\begin{aligned} \mathbb{E} \left[ e^{-\lambda \overline{\mathcal{H}}_1/t} \right] &= \mathbb{E} \left[ e^{-\lambda e_1 S_1^t R_1^t/t} | e_1 S_1^t \leq t/y_t \right] = (1 - p_t)^{-1} \mathbb{E} \left[ e^{-\lambda e_1 S_1^t R_1^t/t} \mathbb{1}_{e_1 S_1^t \leq t/y_t} \right] \\ &= (1 - p_t)^{-1} \int_0^{+\infty} \mathbb{E} \left[ e^{-\lambda u e_1 S_1^t/t} \mathbb{1}_{e_1 S_1^t \leq t/y_t} \right] \mathcal{L}(R_1^t)(du) \\ &= (1 - p_t)^{-1} \int_0^{+\infty} \left( 1 - e^{-\lambda u/y_t} p_t - \lambda \int_0^{1/y_t} u e^{-\lambda x u} \mathbb{P}(e_1 S_1^t/t > x) dx \right) \mathcal{L}(R_1^t)(du), \end{aligned}$$

where we used iteration by parts,

$$= (1 - p_t)^{-1} \left( 1 - \mathbb{E} \left[ e^{-\lambda R_1^t / y_t} \right] p_t - \lambda \int_0^{1/y_t} \mathbb{E} \left[ R_1^t e^{-\lambda x R_1^t} \right] \mathbb{P}(e_1 S_1^t / t > x) dx \right),$$

where we used Fubini's Theorem,

$$= (1 - p_t)^{-1} \left( 1 - p_t + p_t(1 - \mathbb{E} \left[ e^{-\lambda R_1^t / y_t} \right]) - \lambda \int_0^{1/y_t} \mathbb{E} \left[ R_1^t e^{-\lambda x R_1^t} \right] \mathbb{P}(e_1 S_1^t / t > x) dx \right). \quad (2.73)$$

We now study the second and third term in (2.73). Using the fact that the difference between two points of a continuously differentiable function is the integral of its derivative, the last part of Fact 2.8 and the equivalent for  $p_t$  we get

$$p_t(1 - \mathbb{E} \left[ e^{-\lambda R_1^t / y_t} \right]) = p_t \frac{\lambda}{y_t} \int_0^1 \mathbb{E} \left[ R_1^t e^{-\lambda u R_1^t / y_t} \right] du \underset{t \rightarrow +\infty}{\sim} \lambda \frac{p_t \mathbb{E}[\mathcal{R}]}{y_t} \underset{t \rightarrow +\infty}{\sim} \lambda \frac{\mathcal{C}' \mathbb{E}[\mathcal{R}]}{e^{\kappa \phi(t)} y_t^{1-\kappa}}. \quad (2.74)$$

Then, from the last part of Fact 2.8 again,

$$\int_0^{1/y_t} \mathbb{E} \left[ R_1^t e^{-\lambda x R_1^t} \right] \mathbb{P}(e_1 S_1^t / t > x) dx \underset{t \rightarrow +\infty}{\sim} \mathbb{E}[\mathcal{R}] \int_0^{1/y_t} \mathbb{P}(e_1 S_1^t / t > x) dx. \quad (2.75)$$

Recall that  $\eta \in ]0, 1/3[$ .  $\int_0^{1/y_t} \mathbb{P}(e_1 S_1^t / t > x) dx$  equals

$$\begin{aligned} & \int_0^{e^{-(1-2\eta)\phi(t)}} \mathbb{P}(e_1 S_1^t / t > x) dx + e^{-\kappa \phi(t)} \int_{e^{-(1-2\eta)\phi(t)}}^{1/y_t} x^{-\kappa} x^\kappa e^{\kappa \phi(t)} \mathbb{P}(e_1 S_1^t / t > x) dx \\ &= \int_0^{e^{-(1-2\eta)\phi(t)}} \mathbb{P}(e_1 S_1^t / t > x) dx + e^{-\kappa \phi(t)} \int_{e^{-(1-2\eta)\phi(t)}}^{1/y_t} x^{-\kappa} \left( x^\kappa e^{\kappa \phi(t)} \mathbb{P}(e_1 S_1^t / t > x) - \mathcal{C}' \right) dx \\ &+ \mathcal{C}' e^{-\kappa \phi(t)} \int_0^{1/y_t} x^{-\kappa} dx - \mathcal{C}' e^{-\kappa \phi(t)} \int_0^{e^{-(1-2\eta)\phi(t)}} x^{-\kappa} dx. \end{aligned} \quad (2.76)$$

Now, the absolute values of the first and fourth terms of (2.76) are respectively less than  $e^{-(1-2\eta)\phi(t)}$  and  $\mathcal{C}' e^{-(2\eta\kappa + (1-2\eta)\phi(t))} / (1 - \kappa)$ . In particular, thanks to  $2\eta\kappa + (1 - 2\eta) > \kappa$  (which is trivial) and  $\log(y_t) \ll \phi(t)$ , both are negligible with respect to  $e^{-\kappa \phi(t)} / y_t^{1-\kappa}$ . From (2.37) we also have

$$e^{-\kappa \phi(t)} \int_{e^{-(1-2\eta)\phi(t)}}^{1/y_t} x^{-\kappa} \left( x^\kappa e^{\kappa \phi(t)} \mathbb{P}(e_1 S_1^t / t > x) - \mathcal{C}' \right) dx = \mathcal{O}(e^{-\kappa \phi(t)} / y_t^{1-\kappa}),$$

and the third term of (2.76) equals  $\mathcal{C}' e^{-\kappa \phi(t)} / (1 - \kappa) y_t^{1-\kappa}$ . Combining with (2.75) we get

$$-\lambda \int_0^{1/y_t} \mathbb{E} \left[ R_1^t e^{-\lambda x R_1^t} \right] \mathbb{P}(e_1 S_1^t / t > x) dx \underset{t \rightarrow +\infty}{\sim} -\lambda \frac{\mathcal{C}' \mathbb{E}[\mathcal{R}]}{(1 - \kappa) e^{\kappa \phi(t)} y_t^{1-\kappa}}. \quad (2.77)$$

Putting together (2.74) and (2.77) in (2.73) we obtain (2.72). □

**Lemma 2.19.** Recall the  $\lambda_0$  defined in Fact 2.8. For any  $\lambda \in [0, \lambda_0[$  we have

$$\left( \mathbb{E} \left[ e^{\lambda y_t \overline{\mathcal{H}_1} / t} \right] - 1 \right) / p_t \xrightarrow{t \rightarrow +\infty} 1 - \mathbb{E} \left[ e^{\lambda \mathcal{R}} \right] + \lambda \int_0^1 x^{-\kappa} \mathbb{E} \left[ \mathcal{R} e^{\lambda x \mathcal{R}} \right] dx, \quad (2.78)$$

and this limit is positive for all  $\lambda \in ]0, \lambda_0[$ .

*Proof.* For any  $\lambda \in [0, \lambda_0[$  we have  $\lambda y_t e_1 S_1^t / t \in [0, \lambda_0[$  on the event  $\{e_1 S_1^t / t \leq 1/y_t\}$ , so

$$\begin{aligned} \mathbb{E} \left[ e^{\lambda y_t \overline{\mathcal{H}}_1 / t} \right] &= \mathbb{E} \left[ e^{\lambda y_t e_1 S_1^t R_1^t / t} \mathbb{1}_{e_1 S_1^t \leq t/y_t} \right] = (1 - p_t)^{-1} \mathbb{E} \left[ e^{\lambda y_t e_1 S_1^t R_1^t / t} \mathbb{1}_{e_1 S_1^t \leq t/y_t} \right] \\ &= (1 - p_t)^{-1} \int_0^{+\infty} \mathbb{E} \left[ e^{\lambda y_t u e_1 S_1^t / t} \mathbb{1}_{e_1 S_1^t \leq t/y_t} \right] \mathcal{L}(R_1^t)(du) \\ &= (1 - p_t)^{-1} \int_0^{+\infty} \left( 1 - e^{\lambda y_t u / y_t} p_t + \lambda y_t \int_0^{1/y_t} u e^{\lambda y_t x u} \mathbb{P}(e_1 S_1^t / t > x) dx \right) \mathcal{L}(R_1^t)(du) \\ &= (1 - p_t)^{-1} \int_0^{+\infty} \left( 1 - e^{\lambda u} p_t + \lambda \int_0^1 u e^{\lambda y u} \mathbb{P}(e_1 S_1^t / t > y/y_t) dy \right) \mathcal{L}(R_1^t)(du), \end{aligned}$$

where we used iteration by parts and made the change of variable  $y = y_t x$ ,

$$= (1 - p_t)^{-1} \left( 1 - \mathbb{E} \left[ e^{\lambda R_1^t} \right] p_t + \lambda \int_0^1 \mathbb{E} \left[ R_1^t e^{\lambda y R_1^t} \right] \mathbb{P}(e_1 S_1^t / t > y/y_t) dy \right)$$

where we used Fubini's Theorem,

$$= (1 - p_t)^{-1} \left( 1 - p_t + p_t (1 - \mathbb{E} \left[ e^{\lambda R_1^t} \right]) + \lambda \int_0^1 \mathbb{E} \left[ R_1^t e^{\lambda y R_1^t} \right] \mathbb{P}(e_1 S_1^t / t > y/y_t) dy \right). \quad (2.79)$$

According to (2.39) the second term is equivalent to  $p_t(1 - \mathbb{E}[e^{\lambda \mathcal{R}}])$ . We now study the third term in (2.79). Recall that  $\eta \in ]0, 1/3[$ . This term equals

$$\begin{aligned} &\lambda \int_0^{y_t e^{-(1-2\eta)\phi(t)}} \mathbb{E} \left[ R_1^t e^{\lambda y R_1^t} \right] \mathbb{P}(e_1 S_1^t / t > y/y_t) dy \\ &+ \lambda y_t^\kappa e^{-\kappa\phi(t)} \int_{y_t e^{-(1-2\eta)\phi(t)}}^1 y^{-\kappa} \mathbb{E} \left[ R_1^t e^{\lambda y R_1^t} \right] (y/y_t)^\kappa e^{\kappa\phi(t)} \mathbb{P}(e_1 S_1^t / t > y/y_t) dy \\ &= \lambda \int_0^{y_t e^{-(1-2\eta)\phi(t)}} \mathbb{E} \left[ R_1^t e^{\lambda y R_1^t} \right] \mathbb{P}(e_1 S_1^t / t > y/y_t) dy \\ &+ \lambda y_t^\kappa e^{-\kappa\phi(t)} \int_{y_t e^{-(1-2\eta)\phi(t)}}^1 y^{-\kappa} \mathbb{E} \left[ R_1^t e^{\lambda y R_1^t} \right] \left( (y/y_t)^\kappa e^{\kappa\phi(t)} \mathbb{P}(e_1 S_1^t / t > y/y_t) - C' \right) dy \\ &+ \lambda C' y_t^\kappa e^{-\kappa\phi(t)} \int_0^1 y^{-\kappa} \mathbb{E} \left[ R_1^t e^{\lambda y R_1^t} \right] dy - \lambda C' y_t^\kappa e^{-\kappa\phi(t)} \int_0^{y_t e^{-(1-2\eta)\phi(t)}} y^{-\kappa} \mathbb{E} \left[ R_1^t e^{\lambda y R_1^t} \right] dy. \quad (2.80) \end{aligned}$$

Now, thanks to (2.39), the absolute values of the first and fourth terms of (2.80) are ultimately less than  $2\lambda \mathbb{E}[\mathcal{R}] y_t e^{-(1-2\eta)\phi(t)}$  and  $2\lambda C' \mathbb{E}[\mathcal{R}] y_t e^{-(2\eta\kappa + (1-2\eta)\phi(t))} / (1 - \kappa)$ . In particular, thanks to  $2\eta\kappa + (1 - 2\eta) > \kappa$  (which is trivial) and  $\log(y_t) \ll \phi(t)$ , both are negligible with respect to  $e^{-\kappa\phi(t)} y_t^\kappa$ . From (2.37) and (2.39) we also have

$$\lambda y_t^\kappa e^{-\kappa\phi(t)} \int_{y_t e^{-(1-2\eta)\phi(t)}}^1 y^{-\kappa} \mathbb{E} \left[ R_1^t e^{\lambda y R_1^t} \right] \left( (y/y_t)^\kappa e^{\kappa\phi(t)} \mathbb{P}(e_1 S_1^t / t > y/y_t) - C' \right) dy = \mathcal{O}(e^{-\kappa\phi(t)} y_t^\kappa),$$

and, thanks to (2.39), the third term of (2.80) is equivalent to  $\lambda C' y_t^\kappa e^{-\kappa\phi(t)} \int_0^1 y^{-\kappa} \mathbb{E}[\mathcal{R} e^{\lambda y \mathcal{R}}] dy$ . We thus get

$$\begin{aligned} \lambda \int_0^1 \mathbb{E} \left[ R_1^t e^{\lambda y R_1^t} \right] \mathbb{P}(e_1 S_1^t / t > y/y_t) dy &\underset{t \rightarrow +\infty}{\sim} \lambda C' y_t^\kappa e^{-\kappa\phi(t)} \int_0^1 y^{-\kappa} \mathbb{E}[\mathcal{R} e^{\lambda y \mathcal{R}}] dy \\ &\underset{t \rightarrow +\infty}{\sim} \lambda p_t \int_0^1 y^{-\kappa} \mathbb{E}[\mathcal{R} e^{\lambda y \mathcal{R}}] dy. \end{aligned}$$

Putting into (2.79) we obtain (2.78).

We justify the positivity of the limit as follows : we see that the right hand side of (2.78) is equivalent to  $\lambda\kappa\mathbb{E}[\mathcal{R}]/(1-\kappa)$  when  $\lambda$  goes to 0. The limit in (2.78) is therefore positive for small  $\lambda$ . On the other hand,  $(\mathbb{E}[e^{\lambda y_t \overline{\mathcal{H}}_1/t}] - 1)/p_t$  increases with  $\lambda$  so the limit in (2.78) is non-decreasing on  $[0, \lambda_0[$ . We thus get the positivity of the limit for all  $\lambda \in ]0, \lambda_0[$ .

□

We can now study the lower and upper bounds given by Proposition 2.17 :

**Proposition 2.20.** *Let  $y_t$  be chosen as before (that is,  $y_t \rightarrow +\infty$  and  $\log(y_t) \ll \phi(t)$ ). There is a positive constant  $L$  (not depending on the choice of  $y_t$ ), such that for any  $b > 0$ ,  $u > 1$  and  $t$  large enough we have*

$$e^{-ub(1-\kappa)y_t/\kappa\mathbb{E}[\mathcal{R}]} \leq \mathbb{P}\left(Y_1^{b,t}\left(Y_2^{-1,t}(b)\right) \leq 1/y_t\right) \leq e^{-Lby_t}.$$

*Proof. Lower bound*

Let us fix  $\alpha \in ]b(1-\kappa)/\kappa\mathbb{E}[\mathcal{R}], ub(1-\kappa)/\kappa\mathbb{E}[\mathcal{R}][$ . For any  $z > 0$ ,  $k^t(z)$  still denotes the first index  $k \geq 1$  such that  $e_k S_k^t/t \geq z$ . We have

$$\left\{Y_1^{b,t}\left(Y_2^{-1,t}(b)\right) \leq 1/y_t\right\} = \left\{k^t(1/y_t) > \mathcal{N}_{tb}\right\} = \left\{\sum_{i=1}^{k^t(1/y_t)-1} e_i S_i^t R_i^t > tb\right\}. \quad (2.81)$$

Now, recall that  $(\overline{\mathcal{H}}_i)_{i \geq 1}$  are iid random variables such that  $\mathcal{L}(\overline{\mathcal{H}}_1) = \mathcal{L}(e_1 S_1^t R_1^t | e_1 S_1^t \leq t/y_t)$  and let  $T$  be a geometric random variable with parameter  $p_t = \mathbb{P}(e_1 S_1^t > t/y_t)$ , independent from the sequence  $(\overline{\mathcal{H}}_i)_{i \geq 1}$ . Recall that  $p_t \sim C' e^{-\kappa\phi(t)} y_t^\kappa$ . We have

$$\mathbb{P}\left(\sum_{i=1}^{k^t(1/y_t)-1} e_i S_i^t R_i^t > tb\right) = \mathbb{P}\left(\sum_{i=1}^{T-1} \overline{\mathcal{H}}_i > tb\right) \geq \mathbb{P}(T > \alpha h(t) + 1) \times \mathbb{P}\left(\sum_{i=1}^{\lfloor \alpha h(t) \rfloor} \overline{\mathcal{H}}_i > tb\right) \quad (2.82)$$

where we put  $h(t) := y_t/p_t \sim e^{\kappa\phi(t)} y_t^{1-\kappa}/C'$ . We give a lower bound for the two factors in the left hand side of (2.82). We first study the Laplace transform of the normalized sum of the second factor to prove its convergence to a constant number. For any  $\lambda \geq 0$ , we have

$$\mathbb{E}\left[e^{-\lambda \sum_{i=1}^{\lfloor \alpha h(t) \rfloor} \overline{\mathcal{H}}_i/t}\right] = \left(\mathbb{E}\left[e^{-\lambda \overline{\mathcal{H}}_1/t}\right]\right)^{\lfloor \alpha h(t) \rfloor} = e^{\lfloor \alpha h(t) \rfloor \log(1 + \mathbb{E}[e^{-\lambda \overline{\mathcal{H}}_1/t}] - 1)}.$$

According to Lemma 2.18, the exponent is equivalent to

$$-\lambda C' \kappa \mathbb{E}[\mathcal{R}] \alpha h(t)/(1-\kappa) e^{\kappa\phi(t)} y_t^{1-\kappa},$$

and since  $h(t) \sim e^{\kappa\phi(t)} y_t^{1-\kappa}/C'$  we get

$$\mathbb{E}\left[e^{-\lambda \sum_{i=1}^{\lfloor \alpha h(t) \rfloor} \overline{\mathcal{H}}_i/t}\right] \xrightarrow[t \rightarrow +\infty]{} e^{-\lambda \alpha \kappa \mathbb{E}[\mathcal{R}]/(1-\kappa)},$$

so  $\sum_{i=1}^{\lfloor \alpha h(t) \rfloor} \overline{\mathcal{H}}_i/t$  converges in probability to  $\alpha \kappa \mathbb{E}[\mathcal{R}]/(1-\kappa)$  which yields

$$\mathbb{P}\left(\sum_{i=1}^{\lfloor \alpha h(t) \rfloor} \overline{\mathcal{H}}_i > tb\right) \xrightarrow[t \rightarrow +\infty]{} 1, \quad (2.83)$$

since  $\alpha > b(1-\kappa)/\kappa\mathbb{E}[\mathcal{R}]$ .

We now study the first factors in the left hand side of (2.82). Since  $T$  is geometric with parameter  $p_t$ , we have

$$\mathbb{P}(T > \alpha h(t) + 1) = (1 - p_t)^{\lfloor \alpha h(t) + 1 \rfloor} = e^{\lfloor \alpha h(t) + 1 \rfloor \log(1 - p_t)}.$$

Now, since  $h(t) \sim e^{\kappa\phi(t)} y_t^{1-\kappa} / \mathcal{C}'$  and  $p_t \sim \mathcal{C}' e^{-\kappa\phi(t)} y_t^\kappa$  we get

$$\log(\mathbb{P}(T > \alpha h(t) + 1)) \underset{t \rightarrow +\infty}{\sim} \alpha y_t. \quad (2.84)$$

Now, putting (2.83) and (2.84) into (2.82), and combining the latter with (2.81), we get the result for  $t$  large enough since  $\alpha < ub(1 - \kappa)/\kappa\mathbb{E}[\mathcal{R}]$ .

#### Upper bound

Recall (2.81) and the definitions of  $(\overline{\mathcal{H}}_i)_{i \geq 1}$ ,  $p_t$ ,  $T$  and  $h(t)$ . Let us fix  $\alpha > 0$  that will be chosen latter. We have

$$\mathbb{P}\left(\sum_{i=1}^{k^t(t/y_t)-1} e_i S_i^t R_i^t > tb\right) = \mathbb{P}\left(\sum_{i=1}^{T-1} \overline{\mathcal{H}}_i > tb\right) \leq \mathbb{P}(T > \alpha h(t)) + \mathbb{P}\left(\sum_{i=1}^{\lfloor \alpha h(t) \rfloor} \overline{\mathcal{H}}_i > tb\right). \quad (2.85)$$

Let us choose  $\lambda \in ]0, \lambda_0[$  where  $\lambda_0$  defined in Fact 2.8. The second term equals

$$\begin{aligned} \mathbb{P}\left(\sum_{i=1}^{\lfloor \alpha h(t) \rfloor} y_t \overline{\mathcal{H}}_i / t > by_t\right) &\leq e^{-\lambda by_t} \mathbb{E}\left[\exp\left(\lambda \sum_{i=1}^{\lfloor \alpha h(t) \rfloor} y_t \overline{\mathcal{H}}_i / t\right)\right] \\ &= e^{-\lambda by_t} (1 + \mathbb{E}[\exp(\lambda y_t \overline{\mathcal{H}}_1 / t)] - 1)^{\lfloor \alpha h(t) \rfloor} \\ &= e^{-\lambda by_t + \lfloor \alpha h(t) \rfloor \log(1 + \mathbb{E}[\exp(\lambda y_t \overline{\mathcal{H}}_1 / t)] - 1)}. \end{aligned}$$

According to Lemma 2.19, the fact that  $h(t) \sim e^{\kappa\phi(t)} y_t^{1-\kappa} / \mathcal{C}'$  and  $p_t \sim \mathcal{C}' e^{-\kappa\phi(t)} y_t^\kappa$  we have

$$\lfloor \alpha h(t) \rfloor \log(1 + \mathbb{E}[\exp(\lambda y_t \overline{\mathcal{H}}_1 / t)] - 1) \underset{t \rightarrow +\infty}{\sim} \alpha y_t \left(1 - \mathbb{E}[e^{\lambda \mathcal{R}}] + \lambda \int_0^1 x^{-\kappa} \mathbb{E}[\mathcal{R} e^{\lambda x \mathcal{R}}] dx\right).$$

Thanks to the positivity of the limit in Lemma 2.19 we can choose  $\alpha$  such that  $0 < \alpha < b\lambda/2(1 - \mathbb{E}[e^{\lambda \mathcal{R}}] + \lambda \int_0^1 x^{-\kappa} \mathbb{E}[\mathcal{R} e^{\lambda x \mathcal{R}}] dx)$ . We thus get for  $t$  large enough

$$\mathbb{P}\left(\sum_{i=1}^{\lfloor \alpha h(t) \rfloor} y_t \overline{\mathcal{H}}_i / t > by_t\right) \leq e^{-\lambda by_t/2}. \quad (2.86)$$

Since  $T$  is geometric with parameter  $p_t$ , we have

$$\mathbb{P}(T > \alpha h(t)) = (1 - p_t)^{\lfloor \alpha h(t) \rfloor} = e^{\lfloor \alpha h(t) \rfloor \log(1-p_t)} \underset{t \rightarrow +\infty}{\approx} e^{-\alpha y_t} \leq e^{-\alpha y_t/2}, \quad (2.87)$$

where we used the equivalents for  $h(t)$  and  $p_t$  and where the last inequality holds for  $t$  large enough.

Now, putting (2.86) and (2.87) into (2.85), and combining the latter with (2.81), we get the result for  $t$  large enough. □

Fix  $\theta > 1$ . We apply Proposition 2.17 with  $a = \theta^{1/3}$  and Proposition 2.20 (the lower bound with  $b = \theta^{1/3}$ ,  $u = \theta^{1/3}$ ,  $y_t = \theta^{1/3} x_t$  and the upper bound with  $b = 1/4$ ,  $y_t = x_t/2$ ). We get :

**Proposition 2.21.** *Let  $\tilde{L} \in ]0, \min\{L/8, \lambda_0/8\}[$  where  $L$  is the positive constant defined in Proposition 2.20 and  $\lambda_0$  is defined in Fact 2.8. There is a positive constant  $c$  such that for any  $\theta > 1$  and  $t$  large enough we have*

$$e^{-\theta(1-\kappa)x_t/\kappa\mathbb{E}[\mathcal{R}]} - e^{-c\phi(t)} \leq \mathbb{P}(\mathcal{L}_X^*(t) \leq t/x_t) \leq e^{-\tilde{L}x_t} + e^{-c\phi(t)}.$$

We can now prove Theorem 1.9.

*Proof.* of Theorem 1.9

Recall that  $\tilde{L}$  is the positive constant defined in Proposition 2.21 and let  $x_t := 4\log(\log(t))/2\tilde{L}$ . Note that such a choice of  $x_t$  satisfies (2.27) with  $\mu = 2$  and  $D = 4/2\tilde{L}$ . We define the events

$$\mathcal{A}_n := \left\{ \inf_{t \in [2^n, 2^{n+1}]} \frac{\mathcal{L}_X^*(t)}{t/\log(\log(t))} \leq \tilde{L}/4 \right\}.$$

From the increase of  $\mathcal{L}_X^*(\cdot)$  and Proposition 2.21 (the upper bound applied with  $t = 2^n$ ) we have for  $n$  large enough,

$$\begin{aligned} \mathbb{P}(\mathcal{A}_n) &\leq \mathbb{P}\left(\mathcal{L}_X^*(2^n) \leq 2^{n+1}\tilde{L}/4\log(\log(2^n))\right) \leq \exp(-2\log(\log(2^n))) + e^{-c\phi(2^n)} \\ &= (\log(2))^{-2}n^{-2} + e^{-c\phi(2^n)}. \end{aligned}$$

Since  $e^{-c\phi(2^n)} = e^{-c(\log \log(2^n))^2} \leq n^{-2}$  for  $n$  large enough, the above is the general term of a converging series so, using the Borel-Cantelli lemma we deduce that  $\mathbb{P}$ -almost surely,

$$\liminf_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t/\log(\log(t))} \geq \tilde{L}/4,$$

so the  $\liminf$  is positive.

We now prove the upper bound for the  $\liminf$ . Let  $a > 1$  and let  $t_n, u_n, v_n$  (defined from this  $a$ ) and  $X^n$  be as in Subsection 2.2 (recall that  $X^n - v_n$  is equal in law to  $X$  under the annealed probability  $\mathbb{P}$ ). We define

$$\mathcal{B}_n := \left\{ \frac{\mathcal{L}_{X^n}^*(t_n)}{t_n/\log(\log(t_n))} \leq \frac{a^2(1-\kappa)}{\kappa\mathbb{E}[\mathcal{R}]} \right\}, \quad \mathcal{B}'_n := \left\{ \frac{\mathcal{L}_X^*(H(v_n))}{t_n/\log(\log(t_n))} \leq \frac{1}{n} \right\}.$$

We also define  $\mathcal{C}_n$  and  $\mathcal{D}_n$  to be as in Lemma 2.10 and  $\mathcal{E}_n := \mathcal{B}_n \cap \mathcal{C}_n$ . We define  $x_t := \kappa\mathbb{E}[\mathcal{R}]\log(\log(t))/a^2(1-\kappa)$ . Note that such a choice of  $x_t$  satisfies (2.27) with  $\mu = 2$  and  $D = \kappa\mathbb{E}[\mathcal{R}]/a^2(1-\kappa)$ . Recall the notation  $\mathcal{L}_X^{*,+}$  defined in Subsection 1.3. Let us choose  $\eta$  and  $C$  as in Lemma 3.2 of the next section and  $Q$  be as defined in the next section. According to (2.43) and Lemma 3.2 applied with  $u = t_n/n\log(\log(t_n))$ ,  $v = v_n$  we get for all  $n$  large enough,

$$\begin{aligned} \mathbb{P}(\overline{\mathcal{B}'_n}) &\leq \mathbb{P}\left(\inf_{\cdot \in [-\infty, 0]} \mathcal{L}_X(+\infty, \cdot) > t_n/n\log(\log(t_n))\right) + \mathbb{P}\left(\frac{\mathcal{L}_X^{*,+}(H(v_n))}{t_n/\log(\log(t_n))} > \frac{1}{n}\right) \\ &\leq 3(t_n/n\log(\log(t_n)))^{-\kappa/(2+\kappa)} + C\left(v_n/Q + v_n^{7/8}\right)(n\log(\log(t_n)))^\kappa/(t_n)^\kappa + v_n^{-\eta}. \end{aligned}$$

From the definition of  $t_n$ , the fact that  $n^\kappa v_n/t_n^\kappa = n^\kappa e^{-\kappa 2an^{a-1}/3}$  and  $\log(\log(t_n)) = a\log(n)$ , we get

$$\sum_{n \geq 1} \mathbb{P}(\overline{\mathcal{B}'_n}) < +\infty. \quad (2.88)$$

According to Proposition 2.21 (the lower bound applied with  $t = t_n$ ,  $\theta = a$ ) and the definition of  $t_n$  we have

$$\mathbb{P}(\mathcal{B}_n) \geq e^{-\log(\log(t_n))/a} - e^{-c\phi(t_n)} = e^{-a\log(n)/a} - e^{-c\phi(t_n)} = n^{-1} - e^{-c\phi(t_n)}.$$

Since  $e^{-c\phi(t_n)} = e^{-c(\log \log(t_n))^2} \leq n^{-2}$  for  $n$  large enough, we get

$$\sum_{n \geq 1} \mathbb{P}(\mathcal{B}_n) = +\infty. \quad (2.89)$$

Then, the combination of (2.89) and (2.44) yields

$$\sum_{n \geq 1} \mathbb{P}(\mathcal{E}_n) \geq \sum_{n \geq 1} \mathbb{P}(\mathcal{B}_n) - \sum_{n \geq 1} \mathbb{P}(\overline{\mathcal{C}_n}) = +\infty. \quad (2.90)$$

As in the proof of Theorem 1.2, we see that each event  $\mathcal{E}_n$  belongs to the  $\sigma$ -field  $\sigma(V(s) - V(u_n), u_n \leq s \leq u_{n+1}, X(t), H(v_n) \leq t \leq H(v_n) + T_n)$  so the events  $(\mathcal{E}_n)_{n \geq 1}$  are independent. Combining this independence with (2.90) and the Borel-Cantelli Lemma we get that  $\mathbb{P}$ -almost surely, the event  $\mathcal{E}_n$  is realized infinitely many often. Combining with (2.88) and the Borel-Cantelli Lemma we get that  $\mathbb{P}$ -almost surely, the event  $\mathcal{B}'_n \cap \mathcal{B}_n \cap \mathcal{C}_n$  is realized infinitely many often. For  $n$  such that this event is realized we have

$$\frac{\mathcal{L}_X^*(H(v_n) + t_n)}{t_n / \log(\log(t_n))} \leq \frac{\mathcal{L}_{X^n}^*(t_n)}{t_n / \log(\log(t_n))} + \frac{\mathcal{L}_X^*(H(v_n))}{t_n / \log(\log(t_n))} \leq \frac{a^2(1 - \kappa)}{\kappa \mathbb{E}[\mathcal{R}]} + \frac{1}{n}. \quad (2.91)$$

Recall that according to (2.45) and the Borel-Cantelli Lemma we have  $\mathbb{P}$ -almost surely

$$H(v_n) + t_n \underset{n \rightarrow +\infty}{\sim} t_n,$$

so combining with (2.91) we deduce that  $\mathbb{P}$ -almost surely,

$$\liminf_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t / \log(\log(t))} \leq \frac{a^2(1 - \kappa)}{\kappa \mathbb{E}[\mathcal{R}]},$$

and letting  $a$  go to 1 we get the asserted upper bound for the  $\liminf$ . □

### 3. ALMOST SURE BEHAVIOR WHEN $\kappa > 1$

In this section we prove Theorems 1.10 and 1.11. Let us first recall some facts and notations from [12] and [14]. Our proof is based on the study of the so-called generalized Ornstein-Uhlenbeck process defined by

$$Z(x) := e^{V(x)} R \left( \int_0^x e^{-V(y)} dy \right),$$

where  $R$  is a two-dimensional squared Bessel process independent from  $V$ . Let  $L$  be the local time of  $Z$  for the position 1,  $n$  the associated excursion measure, and  $L^{-1}$  the right continuous inverse of  $L$ . We denote by  $\xi$  a generic excursion. Let us denote by  $Q$  the positive constant denoted by  $n[\zeta]$  in [12]. Recall also the notations  $K$  and  $m$  defined in the Introduction. We have :

**Fact 3.1.** *There is  $\eta > 0$  and  $r_0 > 0$  such that for all  $r \geq r_0$  and  $h > 1$  we have*

$$e^{-(r/Q + r^{7/8})n(\sup \xi > h)} - r^{-\eta} \leq \mathbb{P} \left( \sup_{x \in [0, r]} Z(x) \leq h \right) \leq e^{-(r/Q - r^{7/8})n(\sup \xi > h)} + r^{-\eta}, \quad (3.92)$$

$$n(\sup \xi > h) \underset{h \rightarrow +\infty}{\sim} Q 2^\kappa \Gamma(\kappa) \kappa^2 K / h^\kappa. \quad (3.93)$$

The first point is Lemma 2.3 of [14] while the second point is Proposition 5.1 of [12]. Note that Fact 3.1 is true for a general positive  $\kappa$  and not only for  $\kappa > 1$ .

Let us recall the link between  $Z$  and the local time until the hitting times. The local time at point  $x$  and within the hitting time  $H(r)$  is given by :

$$\mathcal{L}_X(H(r), x) = e^{-V(x)} \mathcal{L}_B(\tau(B, A_V(r)), A_V(x)). \quad (3.94)$$



$\mathcal{M}_1(r)$  and  $\mathcal{M}_2(r)$  denote respectively the supremum of the above expression for  $x \in ]-\infty, 0[$  and  $x \in [0, +\infty[$ . The supremum of the local time until instant  $H(r)$  can be written

$$\mathcal{L}_X^*(H(r)) = \max\{\mathcal{M}_1(r), \mathcal{M}_2(r)\}, \quad (3.95)$$

where

$$\mathcal{M}_1(r) \leq \mathcal{M}_1(+\infty) < +\infty \text{ and, as in [14], } \mathcal{M}_2(r) \stackrel{\mathcal{L}}{=} \sup_{x \in [0, r]} Z(x). \quad (3.96)$$

We can now study the behavior of the local time at a hitting time. This allows to prove the following useful lemma.

**Lemma 3.2.** *There exist  $\eta > 0$ , a positive constant  $C$ ,  $u_0 > 0$  and  $v_0 > 0$  such that*

$$\forall u \geq u_0, v \geq v_0, \mathbb{P}\left(\mathcal{L}_X^{*,+}(H(v)) > u\right) \leq C(v/Q + v^{7/8})u^{-\kappa} + v^{-\eta},$$

where  $\mathcal{L}_X^{*,+}$  is as defined in Subsection 1.3.

*Proof.* From the definition of  $\mathcal{M}_2$  and (3.96) we have

$$\mathbb{P}\left(\mathcal{L}_X^{*,+}(H(v)) \geq u\right) = \mathbb{P}(\mathcal{M}_2(v) > u) = \mathbb{P}\left(\sup_{[0, v]} Z > u\right). \quad (3.97)$$

Now let us choose  $\eta$  and  $v_0$  such that (3.92) is true for all  $r \geq v_0$  and  $h > 1$  with this  $\eta$ . We choose  $C > Q2^\kappa \Gamma(\kappa) \kappa^2 K$  and  $u_0$  such that  $n(\sup \xi > u) \leq Cu^{-\kappa}$  for all  $u \geq u_0$ . Such a  $u_0$  exists thanks to (3.93). For  $u \geq u_0$  and  $v \geq v_0$  we have

$$\begin{aligned} \mathbb{P}\left(\sup_{[0, v]} Z > u\right) &\leq 1 - e^{-(v/Q + v^{7/8}) \times n(\sup \xi > u)} + v^{-\eta} \\ &\leq (v/Q + v^{7/8}) \times n(\sup \xi > u) + v^{-\eta} \\ &\leq C(v/Q + v^{7/8})u^{-\kappa} + v^{-\eta}. \end{aligned}$$

Putting into (3.97) we get the result. □

**Remark 3.3.** *Neither Lemma 3.2 nor its proof require the hypothesis that  $\kappa > 1$ . The lemma is thus true whatever is the value of  $\kappa$ .*

We need to study the supremum of the local time until a deterministic time. The following fact from [14] says that we can replace a deterministic time by a hitting time when  $\kappa > 1$ . **We now assume  $\kappa > 1$  until the end of this section.** We have :

**Fact 3.4.** *For any  $\alpha \in ]\max\{3/4, 1/\kappa\}, 1[$  there exists  $\eta > 0$  such that for  $r$  large enough we have*

$$\mathbb{P}(H(r/m - r^\alpha) \leq r \leq H(r/m + r^\alpha)) \geq 1 - r^{-\eta}. \quad (3.98)$$

Let  $\alpha \in ]\max\{3/4, 1/\kappa\}, 1[$  be fixed until the end of this section and  $\eta > 0$  be small enough so that both (3.92) and (3.98) are satisfied (with this  $\alpha$ ).

3.1. **The lim inf.** In this subsection we prove Theorem 1.11. Let us define  $J := 2(\Gamma(\kappa)\kappa^2 K/m)^{1/\kappa}$ , the expected lim inf. We begin to prove that

$$\liminf_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{(t/\log(\log(t)))^{1/\kappa}} \geq J. \quad (3.99)$$

Let  $a > 1$  and define the events

$$\mathcal{A}_n := \left\{ \inf_{t \in [a^n, a^{n+1}]} \frac{\mathcal{L}_X^*(t)}{(t/\log(\log(t)))^{1/\kappa}} \leq \frac{J}{a^{3/\kappa}} \right\}.$$

From the increase of  $\mathcal{L}_X^*(\cdot)$ , (3.98), (3.95), (3.96), and (3.92), we have

$$\begin{aligned} \mathbb{P}(\mathcal{A}_n) &\leq \mathbb{P}\left(\mathcal{L}_X^*(a^n) \leq J(a^{n-2}/\log(\log(a^n)))^{1/\kappa}\right) \\ &\leq \mathbb{P}\left(\mathcal{L}_X^*(H(a^n/m - a^{\alpha n})) \leq J(a^{n-2}/\log(\log(a^n)))^{1/\kappa}\right) + a^{-n\eta} \\ &\leq \mathbb{P}\left(\mathcal{M}_2(a^n/m - a^{\alpha n}) \leq J(a^{n-2}/\log(\log(a^n)))^{1/\kappa}\right) + a^{-n\eta} \\ &= \mathbb{P}\left(\sup_{[0, a^n/m - a^{\alpha n}]} Z \leq J(a^{n-2}/\log(\log(a^n)))^{1/\kappa}\right) + a^{-n\eta} \\ &\leq e^{-\left(a^n/Qm - a^{\alpha n}/Q - (a^n/m - a^{\alpha n})^{7/8}\right) \times n \left(\sup \xi > J(a^{n-2}/\log(\log(a^n)))^{1/\kappa}\right)} + a^{-n\eta} \\ &\quad + (a^n/m - a^{\alpha n})^{-\eta}. \end{aligned} \quad (3.100)$$

According to the equivalent given by (3.93), the exponent in the above expression is, for  $n$  large enough, less than  $-a \log(\log(a^n))$ , so for such large  $n$ ,

$$e^{-\left(a^n/Qm - a^{\alpha n}/Q - (a^n/m - a^{\alpha n})^{7/8}\right) \times n \left(\sup \xi > J(a^{n-2}/\log(\log(a^n)))^{1/\kappa}\right)} \leq (n \log(a))^{-a}.$$

The other two terms in the right hand side of (3.100) are also general terms of converging series so we obtain,

$$\sum_{n \geq 1} \mathbb{P}(\mathcal{A}_n) < +\infty.$$

According to the Borel-Cantelli lemma we get

$$\liminf_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{(t/\log(\log(t)))^{1/\kappa}} \geq J/a^{3/\kappa},$$

in which we can let  $a$  go to 1 which yields (3.99). We now prove that

$$\liminf_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{(t/\log(\log(t)))^{1/\kappa}} \leq J. \quad (3.101)$$

Let us fix  $a > 0$ ,  $u_n := n^{2n}$ ,  $v_n := u_n/m + u_n^\alpha = n^{2n}/m + n^{2\alpha n}$  and  $X^n := X(H(2v_n) + \cdot)$ , the diffusion shifted by the hitting time of  $2v_n$ . Note that from the Markov property for  $X$  at time  $H(2v_n)$  and the stationarity of the increments of  $V$ ,  $X^n - 2v_n$  is equal in law to  $X$  under the

annealed probability  $\mathbb{P}$ . We take  $n$  so large such that  $2v_n < v_{n+1}$  and define the events

$$\begin{aligned}\mathcal{B}_n &:= \left\{ \frac{\mathcal{L}_X^{*,+}(H(2v_n))}{(u_{n+1}/\log(\log(u_{n+1})))^{1/\kappa}} \leq aJ \right\}, \\ \mathcal{C}_n &:= \left\{ \frac{\mathcal{L}_{X^n}^*(\tau(X^n, v_{n+1}))}{(u_{n+1}/\log(\log(u_{n+1})))^{1/\kappa}} \leq (1+a)J \right\}, \\ \mathcal{D}_n &:= \{\tau(X^n, v_{n+1}) < \tau(X^n, v_n)\}, \\ \mathcal{E}_n &:= \mathcal{C}_n \cap \mathcal{D}_n, \\ \mathcal{F}_n &:= \left\{ \mathcal{L}_X^{*,+}(u_n) \leq \mathcal{L}_X^{*,+}(H(v_n)) \right\}.\end{aligned}$$

Recall that  $\eta > 0$  has been fixed so that (3.92) is satisfied. For this  $\eta$  and for  $C > Q2^\kappa\Gamma(\kappa)\kappa^2K$ , the inequality of Lemma 3.2 is true for  $u$  and  $v$  large enough. According to this lemma applied with  $u = aJ(u_{n+1}/\log(\log(u_{n+1})))^{1/\kappa}$ ,  $v = 2v_n$  we get for all  $n$  large enough,

$$\mathbb{P}(\overline{\mathcal{B}_n}) \leq C \left( 2v_n/Q + (2v_n)^{7/8} \right) \log(\log(u_{n+1}))/J^\kappa a^\kappa u_{n+1} + (2v_n)^{-\eta}.$$

Since, for  $n$  large enough,  $v_n/u_{n+1} \leq 1/mn^2$  and  $\log(\log(u_{n+1})) \sim \log(n)$  we can deduce that

$$\sum_{n \geq 1} \mathbb{P}(\overline{\mathcal{B}_n}) < +\infty. \quad (3.102)$$

From the equality in law between  $X^n - 2v_n$  and  $X$  under  $\mathbb{P}$ , , (3.95), (3.96), (2.43), (3.92)

$$\begin{aligned}\mathbb{P}(\mathcal{C}_n) &= \mathbb{P}\left(\mathcal{L}_X^*(H(v_{n+1} - 2v_n)) \leq (1+a)J(u_{n+1}/\log(\log(u_{n+1})))^{1/\kappa}\right) \\ &\geq \mathbb{P}\left(\mathcal{M}_2(H(v_{n+1} - 2v_n)) \leq (1+a)J(u_{n+1}/\log(\log(u_{n+1})))^{1/\kappa}\right) \\ &= \mathbb{P}\left(\inf_{]-\infty, 0]} \mathcal{L}_X(+\infty, \cdot) > (1+a)J(u_{n+1}/\log(\log(u_{n+1})))^{1/\kappa}\right) \\ &\geq \mathbb{P}\left(\sup_{[0, v_{n+1}-2v_n]} Z \leq (1+a)J(u_{n+1}/\log(\log(u_{n+1})))^{1/\kappa}\right) \\ &= 3((1+a)J)^{-\kappa/(2+\kappa)} \times (u_{n+1}/\log(\log(u_{n+1})))^{-1/(2+\kappa)} \\ &\geq e^{-((v_{n+1}-2v_n)/Q + (v_{n+1}-2v_n)^{7/8}) \times n \left(\sup \xi > (1+a)J(u_{n+1}/\log(\log(u_{n+1})))^{1/\kappa}\right)} \\ &= (v_{n+1} - 2v_n)^{-\eta} - 3((1+a)J)^{-\kappa/(2+\kappa)} \times (u_{n+1}/\log(\log(u_{n+1})))^{-1/(2+\kappa)}. \quad (3.103)\end{aligned}$$

According to the equivalent given by (3.93) and the definitions of  $u_n$  and  $v_n$ , the exponent in the above expression is equivalent to  $(1+a)^{-\kappa} \log(\log(u_{n+1})) \sim (1+a)^{-\kappa} \log(n)$  so for  $n$  large enough,

$$e^{-((v_{n+1}-2v_n)/Q + (v_{n+1}-2v_n)^{7/8}) \times n \left(\sup \xi > (1+a)J(u_{n+1}/\log(\log(u_{n+1})))^{1/\kappa}\right)} \geq \frac{1}{n},$$

and the remaining terms in the right hand side of (3.103) are the general terms of converging series. We thus get

$$\sum_{n \geq 1} \mathbb{P}(\mathcal{C}_n) = +\infty. \quad (3.104)$$

$$\begin{aligned}
\mathbb{P}(\overline{\mathcal{D}_n}) &= \mathbb{P}(\tau(X^n, v_{n+1}) > \tau(X^n, v_n)) \\
&\leq \mathbb{P}(\tau(X^n, v_n) < \tau(X^n, +\infty)) \\
&\leq \mathbb{P}(\tau(X, -v_n) < \tau(X, +\infty)) \\
&= \mathbb{P}\left(\inf_{[0, +\infty[} X < -v_n\right),
\end{aligned}$$

where we used the equality in law between  $X^n - 2v_n$  and  $X$  under  $\mathbb{P}$ . Combining with (2.42) applied with  $r = v_n$  we get

$$\sum_{n \geq 1} \mathbb{P}(\overline{\mathcal{D}_n}) < +\infty. \quad (3.105)$$

Then, the combination of (3.104) and (3.105) yields

$$\sum_{n \geq 1} \mathbb{P}(\mathcal{E}_n) \geq \sum_{n \geq 1} \mathbb{P}(\mathcal{C}_n) - \sum_{n \geq 1} \mathbb{P}(\overline{\mathcal{D}_n}) = +\infty. \quad (3.106)$$

According to the definitions of the sequences  $(u_n)_{n \geq 1}$  and  $(v_n)_{n \geq 1}$ , to (3.98), and to the increase of  $\mathcal{L}_X^{*,+}$ , we have, for  $n$  large enough,  $\mathbb{P}(\mathcal{F}_n) \leq u_n^{-\eta}$ , so

$$\sum_{n \geq 1} \mathbb{P}(\overline{\mathcal{F}_n}) < +\infty. \quad (3.107)$$

Note that each event  $\mathcal{E}_n$  belongs to the  $\sigma$ -field  $\sigma(V(s) - V(v_n), v_n \leq s \leq v_{n+1}, X(t), H(2v_n) \leq t \leq \min(\tau(X^n, v_n), \tau(X^n, v_{n+1})))$ , in other words, it only depends on the diffusion between times  $H(2v_n)$  and  $\min(\tau(X^n, v_n), \tau(X^n, v_{n+1}))$  and on the environment between positions  $v_n$  and  $v_{n+1}$ . From the Markov property and the independence of the increments of the environment, we get that the events  $(\mathcal{E}_n)_{n \geq 1}$  are independent. Combining this independence with (3.106) and the Borel-Cantelli Lemma we get that  $\mathbb{P}$ -almost surely, the event  $\mathcal{E}_n$  is realized infinitely many often.

Combining (3.102), (3.107) and the Borel-Cantelli Lemma we get that  $\mathbb{P}$ -almost surely the event  $\mathcal{B}_n \cap \mathcal{F}_n$  is realized for all large  $n$ . We deduce that  $\mathbb{P}$ -almost surely, the event  $\mathcal{B}_n \cap \mathcal{C}_n \cap \mathcal{D}_n \cap \mathcal{F}_{n+1}$  is realized infinitely many often. Then, for  $n$  such that this event is realized we have,

$$\begin{aligned}
\frac{\mathcal{L}_X^{*,+}(u_{n+1})}{(u_{n+1}/\log(\log(u_{n+1})))^{1/\kappa}} &\leq \frac{\mathcal{L}_X^{*,+}(H(v_{n+1}))}{(u_{n+1}/\log(\log(u_{n+1})))^{1/\kappa}} \\
&\leq \frac{\mathcal{L}_X^{*,+}(H(2v_n))}{(u_{n+1}/\log(\log(u_{n+1})))^{1/\kappa}} + \frac{\mathcal{L}_{X^n}^*(\tau(X^n, v_{n+1}))}{(u_{n+1}/\log(\log(u_{n+1})))^{1/\kappa}} \\
&\leq aJ + (1+a)J,
\end{aligned}$$

so

$$\liminf_{t \rightarrow +\infty} \frac{\mathcal{L}_X^{*,+}(t)}{(t/\log(\log(t)))^{1/\kappa}} \leq (1+2a)J.$$

Now, letting  $a$  go to 0 and combining with the finiteness of  $\sup_{]-\infty, 0[} \mathcal{L}_X(+\infty)$  (see (3.95) and (3.96)) we obtain (3.101) so Theorem 1.11 is proved.

**3.2. The lim sup.** In this subsection we prove Theorem 1.10. First, let us assume that  $\int_1^{+\infty} \frac{(f(t))^\kappa}{t} dt < +\infty$  and prove that

$$\limsup_{t \rightarrow +\infty} \frac{f(t)\mathcal{L}_X^*(t)}{t} = 0. \quad (3.108)$$

According to Remark 1.19 the condition  $\int_1^{+\infty} \frac{(f(t))^\kappa}{t} dt < +\infty$  is equivalent to

$$\sum_{n=1}^{+\infty} (f(2^n))^\kappa < +\infty. \quad (3.109)$$

Let us fix  $a > 0$  and define the events

$$\mathcal{A}_n := \left\{ \sup_{t \in [2^n, 2^{n+1}]} \frac{f(t) \mathcal{L}_X^{*,+}(t)}{t^{1/\kappa}} \geq a \right\}.$$

From the increase of  $\mathcal{L}_X^{*,+}(\cdot)$ , (3.98), (3.95), (3.96), and (3.92), we have

$$\begin{aligned} \mathbb{P}(\mathcal{A}_n) &\leq \mathbb{P}\left(\mathcal{L}_X^{*,+}(2^{n+1}) \geq 2^{n/\kappa} a / f(2^n)\right) \\ &\leq \mathbb{P}\left(\mathcal{L}_X^{*,+}\left(H(2^{n+1}/m + 2^{\alpha(n+1)})\right) \geq 2^{n/\kappa} a / f(2^n)\right) + 2^{-\eta(n+1)} \\ &= \mathbb{P}\left(\mathcal{M}_2\left(2^{n+1}/m + 2^{\alpha(n+1)}\right) \geq 2^{n/\kappa} a / f(2^n)\right) + 2^{-\eta(n+1)} \\ &= \mathbb{P}\left(\sup_{[0, 2^{n+1}/m + 2^{\alpha(n+1)}]} Z \geq 2^{n/\kappa} a / f(2^n)\right) + 2^{-\eta(n+1)} \\ &\leq 1 - e^{-\left(2^{n+1}/Qm + 2^{\alpha(n+1)}/Q + (2^{n+1}/m + 2^{\alpha(n+1)})^{7/8}\right) \times n \left(\sup \xi > 2^{n/\kappa} a / f(2^n)\right)} \\ &\quad + 2^{-\eta(n+1)} + \left(2^{n+1}/m + 2^{\alpha(n+1)}\right)^{-\eta} \\ &\leq \left(2^{n+1}/Qm + 2^{\alpha(n+1)}/Q + \left(2^{n+1}/m + 2^{\alpha(n+1)}\right)^{7/8}\right) \times n \left(\sup \xi > 2^{n/\kappa} a / f(2^n)\right) \\ &\quad + 2^{-\eta(n+1)} + \left(2^{n+1}/m + 2^{\alpha(n+1)}\right)^{-\eta}. \end{aligned}$$

According to (3.93), the first term in the right hand side is equivalent to

$$2^{1+\kappa} \Gamma(\kappa) \kappa^2 K(f(2^n))^\kappa / m a^\kappa$$

which is the general term of a convergent series, according to (3.109). The two remaining terms are also the general terms of convergent series so we get

$$\sum_{n \geq 1} \mathbb{P}(\mathcal{A}_n) < +\infty,$$

and applying the Borel-Cantelli lemma we deduce

$$\limsup_{t \rightarrow +\infty} \frac{f(t) \mathcal{L}_X^{*,+}(t)}{t} \leq a.$$

Now, letting  $a$  go to 0 and combining with the finiteness of  $\sup_{]-\infty, 0[} \mathcal{L}_X(+\infty)$  (see (3.95) and (3.96)) we obtain (3.108).

Let us now assume that  $\int_1^{+\infty} \frac{(f(t))^\kappa}{t} dt = +\infty$  and prove that

$$\limsup_{t \rightarrow +\infty} \frac{f(t) \mathcal{L}_X^*(t)}{t} = +\infty. \quad (3.110)$$

According to Remark 1.19 the condition  $\int_1^{+\infty} \frac{(f(t))^\kappa}{t} dt = +\infty$  is equivalent to

$$\sum_{n=1}^{+\infty} (f(2^n))^\kappa = +\infty. \quad (3.111)$$

Let  $M > 0$ ,  $u_n := 2^n/m - 2^{\alpha n}$  and  $X^n := X(H(\sqrt{2}u_n) + \cdot)$ , the diffusion shifted by the hitting time of  $\sqrt{2}u_n$ . Note that from the Markov property for  $X$  at time  $H(\sqrt{2}u_n)$  and the stationarity of the increments of  $V$ ,  $X^n - \sqrt{2}u_n$  is equal in law to  $X$  under the annealed probability  $\mathbb{P}$ . We take  $n$  so large such that  $\sqrt{2}u_n < u_{n+1}$  and define the events

$$\begin{aligned}\mathcal{C}_n &:= \left\{ \frac{f(2^{n+1})\mathcal{L}_{X^n}^*(\tau(X^n, u_{n+1}))}{2^{(n+1)/\kappa}} \geq M \right\}, \\ \mathcal{D}_n &:= \{\tau(X^n, u_{n+1}) < \tau(X^n, u_n)\}, \\ \mathcal{E}_n &:= \mathcal{C}_n \cap \mathcal{D}_n, \\ \mathcal{F}_n &:= \{\mathcal{L}_X^*(H(u_n)) \leq \mathcal{L}_X^*(2^n)\}.\end{aligned}$$

From the equality in law between  $X^n - \sqrt{2}u_n$  and  $X$  under  $\mathbb{P}$ , (3.95), (3.96), and (3.92), we have

$$\begin{aligned}\mathbb{P}(\mathcal{C}_n) &= \mathbb{P}\left(\mathcal{L}_X(H(u_{n+1} - \sqrt{2}u_n)) \geq 2^{(n+1)/\kappa} M / f(2^{n+1})\right) \\ &\geq \mathbb{P}\left(\mathcal{L}_X^{*,+}(H(u_{n+1} - \sqrt{2}u_n)) \geq 2^{(n+1)/\kappa} M / f(2^{n+1})\right) \\ &= \mathbb{P}\left(\mathcal{M}_2(u_{n+1} - \sqrt{2}u_n) \geq 2^{(n+1)/\kappa} M / f(2^{n+1})\right) \\ &= \mathbb{P}\left(\sup_{[0, u_{n+1} - \sqrt{2}u_n]} Z \geq 2^{(n+1)/\kappa} M / f(2^{n+1})\right) \\ &\geq 1 - e^{-((u_{n+1} - \sqrt{2}u_n)/Q - (u_{n+1} - \sqrt{2}u_n)^{7/8}) \times n (\sup \xi > 2^{(n+1)/\kappa} M / f(2^{n+1}))} \\ &\quad - (u_{n+1} - \sqrt{2}u_n)^{-\eta}.\end{aligned}\tag{3.112}$$

According to (3.93) and the definition of  $u_n$ , the exponent in the right hand side is equivalent to

$$(2 - \sqrt{2})2^{\kappa-1}\Gamma(\kappa)\kappa^2 K(f(2^{n+1}))^\kappa / m M^\kappa.$$

Since  $f$  converges to 0 at infinity, the above is also an equivalent for the term  $1 - e^{-(\dots)}$  in the right hand side of (3.112). Then, combining with (3.111) and the fact that the other term is the general term of a covering series we get

$$\sum_{n \geq 1} \mathbb{P}(\mathcal{C}_n) = +\infty.\tag{3.113}$$

Reasoning as in the proof of (3.105) we can prove that

$$\sum_{n \geq 1} \mathbb{P}(\overline{\mathcal{D}_n}) < +\infty,\tag{3.114}$$

so

$$\sum_{n \geq 1} \mathbb{P}(\mathcal{E}_n) \geq \sum_{n \geq 1} \mathbb{P}(\mathcal{C}_n) - \sum_{n \geq 1} \mathbb{P}(\overline{\mathcal{D}_n}) = +\infty.\tag{3.115}$$

According to (3.98), and to the increase of  $\mathcal{L}_X^*$ , we also prove that

$$\sum_{n \geq 1} \mathbb{P}(\overline{\mathcal{F}_n}) < +\infty.\tag{3.116}$$

The events  $\mathcal{E}_n$  are independent since for each  $n$ ,  $\mathcal{E}_n$  belongs to the  $\sigma$ -field  $\sigma(V(s) - V(u_n), u_n \leq s \leq u_{n+1}, X(t), H(\sqrt{2}u_n) \leq t \leq H(\sqrt{2}u_n) + \min(\tau(X^n, u_n), \tau(X^{n+1}, u_{n+1}))$ . Combining this independence with (3.115) and the Borel-Cantelli Lemma we get that  $\mathbb{P}$ -almost surely, the event

$\mathcal{E}_n$  is realized infinitely many often. Combining with (3.116) and the Borel-Cantelli Lemma we get that  $\mathbb{P}$ -almost surely, the event  $\mathcal{C}_n \cap \mathcal{D}_n \cap \mathcal{F}_{n+1}$  is realized infinitely many often. Then, for  $n$  such that this event is realized we have,

$$\begin{aligned} f(2^{n+1})\mathcal{L}_X^*(2^{n+1})/2^{(n+1)/\kappa} &\geq f(2^{n+1})\mathcal{L}_X^*(H(u_{n+1}))/2^{(n+1)/\kappa} \\ &\geq f(2^{n+1})\mathcal{L}_{X^n}^*(\tau(X^n, u_{n+1}))/2^{(n+1)/\kappa} \\ &\geq M, \end{aligned}$$

so

$$\limsup_{t \rightarrow +\infty} \frac{f(t)\mathcal{L}_X^*(t)}{t} \geq M.$$

Now, letting  $M$  go to infinity we get (3.110) so Theorem 1.10 is proved.

#### 4. SOME LEMMAS

In this section, we justify some technical facts and lemmas for  $V$ ,  $V^\uparrow$  and the diffusion in  $V$ . Some of them are known or can be easily obtained from results of [14], and we give some details for their justification when it is necessary. Some of these facts are new, like the approximation of the contributions of the valleys to the traveled distance by an *iid* sequence.

##### 4.1. Properties of $V$ , $V^\uparrow$ and $\hat{V}^\uparrow$ .

**Lemma 4.1.** (Lemma 5.4 of [14])

There are two positive constants  $c_1, c_2$  such that

$$\forall y, r > 0, P(\tau(V, ] - \infty, -y]) > r) \leq e^{c_1 y - c_2 r}.$$

**Lemma 4.2.** There are positive constants  $c_1, c_2$  such that,

$$\forall t, a > 0, P\left(\sup_{[t, +\infty[} V > -a\right) \leq e^{c_1 a - c_2 t} + e^{-\kappa a}.$$

*Proof.* Let us choose  $\gamma \in ]0, \kappa[$ , we have

$$\begin{aligned} P\left(\sup_{[t, +\infty[} V > -a\right) &\leq P(V(t) > -2a) + P\left(V(t) \leq -2a, \sup_{[0, +\infty[} V(t + \cdot) - V(t) > a\right) \\ &\leq P(e^{\gamma V(t)} > e^{-2\gamma a}) + P\left(\sup_{[0, +\infty[} V(t + \cdot) - V(t) > a\right) \\ &= e^{2\gamma a} E[e^{\gamma V(t)}] + e^{-\kappa a}, \end{aligned}$$

where we used Markov's inequality for the first term and the Markov property at time  $t$  for the second term, together with the fact that the supremum of  $V$  on  $[0, +\infty[$  follows an exponential distribution with parameter  $\kappa$ . Since  $E[e^{\gamma V(t)}] = e^{t\Psi_V(\gamma)}$  and  $\Psi_V(\gamma) < 0$  (because  $0 < \gamma < \kappa$ ), we get the result with  $c_1 := 2\gamma$  and  $c_2 := -\Psi_V(\gamma)$ . □

**Lemma 4.3.** (Lemma 5.3 of [14])

There is a positive constant  $\mathcal{C}$  such that

$$\mathbb{P}\left(\int_0^{+\infty} e^{V(u)} du \geq x\right) \underset{x \rightarrow +\infty}{\sim} \mathcal{C} x^{-\kappa}.$$



We now state some Lemmas about  $V^\uparrow$ . First, we recall how  $V^\uparrow$  and  $\hat{V}^\uparrow$  are defined.

$V$  being spectrally negative, the Markov family  $(V_x^\uparrow, x \geq 0)$  may be defined as in [4], Section VII.3. For any  $x \geq 0$ , the process  $V_x^\uparrow$  must be seen as  $V$  conditioned to stay positive and starting from  $x$ . We denote  $V^\uparrow$  for the process  $V_0^\uparrow$ . It is known that  $V_x^\uparrow$  converges in the Skorokhod space to  $V^\uparrow$  when  $x$  goes to 0. Also, as well as  $V$ ,  $V^\uparrow$  has no positive jumps so it reaches every positive level continuously.

Since  $V$  is spectrally negative and not the opposite of a subordinator, it is regular for  $]0, +\infty[$  (see [4], Theorem VII.1), so  $\hat{V}$  is for  $] - \infty, 0[$ . Moreover,  $\hat{V}$  drifts to  $+\infty$ . We can thus define the Markov family  $(\hat{V}_x^\uparrow, x \geq 0)$  as in Doney [8], Chapter 8. It can be seen from there that the processes such defined are Markov and have infinite life-time. If moreover  $V$  has unbounded variation then  $\hat{V}$  is regular for  $]0, +\infty[$ , and from Theorem 24 of [8], we have that  $\hat{V}_0^\uparrow$ , that we denote by  $\hat{V}^\uparrow$ , is well defined.

Here again, for any  $x \geq 0$ , the process  $\hat{V}_x^\uparrow$  must be seen as  $\hat{V}$  conditioned to stay positive and starting from  $x$ . Note that, since  $\hat{V}$  converges almost surely to infinity, for  $x > 0$ ,  $\hat{V}_x^\uparrow$  is only  $\hat{V}_x$  conditioned in the usual sense to remain positive.

**Lemma 4.4.** (Lemma 5.7 of [14])

There are two positive constants  $c_1, c_2$  such that, for all  $1 < a < b$ , we have

$$\mathbb{P} \left( \inf_{[0, +\infty[} V_b^\uparrow < a \right) \leq c_2 e^{-c_1(b-a)}.$$

**Lemma 4.5.** There are positive constants  $c_3, c_4, c_5, c_6$  such that,

$$\forall 0 \leq x < y, r > 0, \mathbb{P} \left( \tau(V_x^\uparrow, y) > r \right) \leq e^{c_3 y - c_4 r}, \quad (4.117)$$

$$\forall z, r > 0, \mathbb{P} \left( \mathcal{K}(V_z^\uparrow, z) > r \right) \leq e^{2c_3 z - c_4 r} + c_6 e^{-c_5 z}. \quad (4.118)$$

*Proof.* Let us fix  $0 \leq x < y$ . From the Markov property applied at  $\tau(V^\uparrow, x)$ , the hitting time of  $x$  by  $V^\uparrow$ , we have

$$\mathbb{P} \left( \tau(V_x^\uparrow, y) > r \right) = \mathbb{P} \left( \tau(V^\uparrow(\tau(V^\uparrow, x) + \cdot), y) > r \right) \leq \mathbb{P} \left( \tau(V^\uparrow, y) > r \right),$$

so (4.117) follows from Lemma 5.6 of [14]. For the second point, we have

$$\left\{ \tau(V_z^\uparrow, 2z) \leq r \right\} \cap \left\{ \inf_{[\tau(V_z^\uparrow, 2z), +\infty[} V_z^\uparrow > z \right\} \subset \left\{ \mathcal{K}(V_z^\uparrow, z) \leq r \right\},$$

so taking the complementary,

$$\begin{aligned} \mathbb{P} \left( \mathcal{K}(V_z^\uparrow, z) > r \right) &\leq \mathbb{P} \left( \tau(V_z^\uparrow, 2z) > r \right) + \mathbb{P} \left( \inf_{[\tau(V_z^\uparrow, 2z), +\infty[} V_z^\uparrow \leq z \right) \\ &\leq e^{2c_3 z - c_4 r} + \mathbb{P} \left( \inf_{[0, +\infty[} V_{2z}^\uparrow \leq z \right), \end{aligned}$$

where, for the first term, we used (4.117) with  $x = z$ ,  $y = 2z$  and, for the second term, we used the Markov property at time  $\tau(V_z^\uparrow, 2z)$ . Combining with Lemma 4.4 applied with  $a = z$ ,  $b = 2z$ , we get (4.118). □

**Lemma 4.6.** *There is a positive constant  $c$  such that for  $t$  large enough,*

$$\mathbb{P} \left( \int_{\tau(V^\uparrow, h_t/2)}^{+\infty} e^{-V^\uparrow(x)} dx \geq e^{-h_t/4} \right) \leq e^{-ch_t}.$$

*Proof.*

$$\begin{aligned} \int_{\tau(V^\uparrow, h_t/2)}^{\mathcal{K}(V^\uparrow, h_t/2)} e^{-V^\uparrow(x)} dx &\leq \left( \mathcal{K}(V^\uparrow, h_t/2) - \tau(V^\uparrow, h_t/2) \right) \times \sup_{[\tau(V^\uparrow, h_t/2), \mathcal{K}(V^\uparrow, h_t/2)]} e^{-V^\uparrow} \\ &\stackrel{\mathcal{L}}{=} \mathcal{K}(V_{h_t/2}^\uparrow, h_t/2) \times \sup_{[0, \mathcal{K}(V_{h_t/2}^\uparrow, h_t/2)]} e^{-V_{h_t/2}^\uparrow}, \end{aligned}$$

where we used the Markov property at time  $\tau(V^\uparrow, h_t/2)$  for the equality in law. We thus get

$$\begin{aligned} \mathbb{P} \left( \int_{\tau(V^\uparrow, h_t/2)}^{\mathcal{K}(V^\uparrow, h_t/2)} e^{-V^\uparrow(x)} dx \geq e^{-h_t/4}/2 \right) &\leq \mathbb{P} \left( \mathcal{K}(V_{h_t/2}^\uparrow, h_t/2) \geq e^{h_t/8}/2 \right) + \mathbb{P} \left( \inf_{[0, +\infty[} V_{h_t/2}^\uparrow \leq 3h_t/8 \right) \\ &\leq e^{c_3 h_t - c_4 e^{h_t/8}/2} + c_6 e^{-c_5 h_t/2} + c_2 e^{-c_1 h_t/8}, \end{aligned} \quad (4.119)$$

where, for the first term, we applied (4.118) with  $z = h_t/2$ ,  $r = e^{h_t/8}/2$  and, for the second term, we applied Lemma 4.4 with  $a = 3h_t/8$ ,  $b = h_t/2$ .

Then, according to Corollary VII.19 of [4] we have

$$\int_{\mathcal{K}(V^\uparrow, h_t/2)}^{+\infty} e^{-V^\uparrow(x)} dx \stackrel{\mathcal{L}}{=} e^{-h_t/2} \int_0^{+\infty} e^{-V^\uparrow(x)} dx = e^{-h_t/2} I(V^\uparrow),$$

and, according to Theorem 1.1 of [13],  $I(V^\uparrow)$  admits some finite exponential moments, so in particular it has finite expectation. We thus get

$$\mathbb{P} \left( \int_{\mathcal{K}(V^\uparrow, h_t/2)}^{+\infty} e^{-V^\uparrow(x)} dx \geq e^{-h_t/4}/2 \right) = \mathbb{P} \left( I(V^\uparrow) \geq e^{h_t/4}/2 \right) \leq 2e^{-h_t/4} \mathbb{E} [I(V^\uparrow)]. \quad (4.120)$$

The result follows from the combination of (4.119) and (4.120).  $\square$

The next fact gives the law of the bottom of the valleys in terms of the laws of  $V^\uparrow$  and  $\hat{V}^\uparrow$ . It is a combination of Propositions 3.2 and 3.6 of [14].

**Fact 4.7.** *Assume  $V$  has unbounded variation. For all  $i \geq 1$  let*

$$\begin{aligned} P^{(i)} &:= (V^{(i)}(m_i - x), 0 \leq x \leq m_i - \tau_i^-(h_t)) \\ \tilde{P}^{(i)} &:= (\tilde{V}^{(i)}(\tilde{m}_i - x), 0 \leq x \leq \tilde{m}_i - \tilde{\tau}_i^-(h_t)). \end{aligned}$$

*For all  $i \geq 1$  we have*

$$d_{VT} \left( \tilde{P}^{(i)}, P^{(2)} \right) \leq 2e^{-\delta \kappa h_t/3} \quad (4.121)$$

where  $d_{VT}$  is the total variation distance. Moreover, the law of  $P^{(2)}$  is absolutely continuous with respect to the law of the process  $(\hat{V}^\uparrow(x), 0 \leq x \leq \tau(\hat{V}^\uparrow, h_t+))$  and has density  $c_{h_t}/(1 - e^{-\kappa \hat{V}^\uparrow(\tau(\hat{V}^\uparrow, h_t+))})$  with respect to this law, where  $c_{h_t}$  is a constant increasing with  $h_t$  and converging to 1 when  $t$  (and hence  $h_t$ ) goes to infinity.

For all  $i \geq 1$ , the two processes  $(\tilde{V}^{(i)}(\tilde{m}_i - x), 0 \leq x \leq \tilde{m}_i - \tilde{\tau}_i^-(h_t)) = \tilde{P}^{(i)}$  and  $(\tilde{V}^{(i)}(\tilde{m}_i + x), 0 \leq x \leq \tilde{\tau}_i(h_t) - \tilde{m}_i)$  are independent and the second is equal in law to  $(V^\uparrow(x), 0 \leq x \leq \tau(V^\uparrow, h_t))$ .

Let us now recall a fact from [14] :

**Fact 4.8.** *Assume that the hypotheses of Theorems 1.4 and 1.9 are satisfied. Fix  $\epsilon$  small enough. There is a positive constant  $c$  (depending on  $\epsilon$ ) such that for all  $t$  large enough*

$$\forall j \geq 1, P \left( \int_{\tilde{L}_{j-1}}^{\tilde{\tau}_j^-(h_t/2)} e^{-\tilde{V}^{(j)}(u)} du \leq e^{-\epsilon h_t} \right) \geq 1 - e^{-\epsilon h_t}, \quad (4.122)$$

$$\forall j \geq 1, P \left( \int_{\tilde{\tau}_j^+(h_t/2)}^{\tilde{L}_j} e^{-\tilde{V}^{(j)}(u)} du \leq e^{-\epsilon h_t} \right) \geq 1 - e^{-\epsilon h_t}, \quad (4.123)$$

$$\forall j \geq 1, P \left( \sup_{u \in [\tilde{\tau}_j^-(h_t/2), \tilde{\tau}_j^+(h_t/2)]} |A^j(u)/A^j(\tilde{L}_j)| \leq e^{-h_t/3} \right) \geq 1 - e^{-\epsilon h_t}, \quad (4.124)$$

$$\forall j \geq 1, P \left( \sup_{u \in [\tilde{\tau}_j^-(h_t/2), \tilde{\tau}_j^+(h_t/2)]} |A^j(u)| \leq e^{5h_t/8} \right) \geq 1 - e^{-\epsilon h_t}, \quad (4.125)$$

where  $A^j$  is defined in Subsection 2.1.

*Proof.* (4.122), (4.123) and (4.124) are respectively Lemma 4.6, Lemma 4.7 and Lemma 4.8 (applied with  $\epsilon = 1/6$ ) from [14]. For (4.125), note that

$$P \left( \sup_{u \in [\tilde{\tau}_j^-(h_t/2), \tilde{\tau}_j^+(h_t/2)]} |A^j(u)| \leq e^{5h_t/8} \right) \leq P \left( |A^j(\tilde{\tau}_j^-(h_t/2))| \vee |A^j(\tilde{\tau}_j^+(h_t/2))| \geq e^{5h_t/8} \right).$$

Then,  $|A^j(\tilde{\tau}_j^-(h_t/2))|$  and  $|A^j(\tilde{\tau}_j^+(h_t/2))|$  can be bounded as in the proof of Lemma 4.8 of [14] which yields (4.125). □

**4.2. Contribution of the valleys to the traveled distance.** We need to approximate the contribution to the distance traveled by the diffusion in each valley. We have :

**Proposition 4.9.** *On an enlarged probability space, there is an iid sequence  $(D_j^t)_{j \geq 1}$ , independent from the sequence  $(e_j, S_j^t, R_j^t)_{j \geq 1}$  and such that for  $t$  large enough,*

$$\forall j \geq 1, \mathbb{P} \left( |(\tilde{L}_j - \tilde{L}_{j-1}) - D_j^t| \leq e^{-c_1 h_t} D_j^t \right) \geq 1 - e^{-c_2 h_t} \text{ and } D_j^t \stackrel{\mathcal{L}}{\sim} \mathcal{E}(e^{-\kappa h_t} q),$$

where  $q$  is the constant in Theorem 1.4 of [14] and  $c_1, c_2$  are positive constants.

*Proof.* From the definitions of valleys we have

$$\tilde{L}_j - \tilde{L}_{j-1} = (\tilde{L}_j^\# - \tilde{L}_{j-1}) + (\tilde{m}_j - \tilde{L}_j^\#) + (\tilde{\tau}_j(h_t) - \tilde{m}_j) + (\tilde{L}_j - \tilde{\tau}_j(h_t)). \quad (4.126)$$

Recall the definition of  $m^*(h_t)$  from [14] :

$$\tau^*(h_t) := \inf \left\{ u \geq 0, V(u) - \inf_{[0, u]} V = h_t \right\}, \quad m^*(h_t) := \inf \left\{ u \geq 0, V(u) = \inf_{[0, \tau^*(h_t)]} V \right\}.$$

From the Markov property at the stopping times  $\tilde{L}_{j-1}$ ,  $\tilde{L}_j^\#$  and  $\tilde{\tau}_j(h_t)$ , and Fact 4.7 we get that the terms in the right hand side of (4.126) are respectively equal in law to  $\tau(V, ] - \infty, -e^{(1-\delta)\kappa h_t}]$ ,  $m^*(h_t)$ ,  $\tau(V^\uparrow, h_t)$  and  $\tau(V, ] - \infty, -h_t/2]$ . For the first and fourth term, applying Lemma 4.1 with  $y = e^{(1-\delta)\kappa h_t}$  and  $r = e^{(1-\delta/2)\kappa h_t}/2$  we get for  $t$  large enough,

$$\forall j \geq 1, \mathbb{P} \left( (\tilde{L}_j^\# - \tilde{L}_{j-1}) + (\tilde{L}_j - \tilde{\tau}_j(h_t)) > e^{(1-\delta/2)\kappa h_t} \right) \leq e^{-h_t}. \quad (4.127)$$

For the third term, applying (4.117) with  $x = 0$ ,  $y = h_t$  and  $r = e^{(1-\delta/2)\kappa h_t}$  we get for  $t$  large enough,

$$\forall j \geq 1, \mathbb{P} \left( (\tilde{\tau}_j(h_t) - \tilde{m}_j) > e^{(1-\delta/2)\kappa h_t} \right) \leq e^{-h_t}. \quad (4.128)$$

The main term is the second one, its law is given by Lemma 3.9 of [14]. Before studying it, let us recall some of the notations used in [14].

Let  $\mathcal{F}$  denote the space of excursions, that is, càd-làg functions from  $[0, +\infty[$  to  $\mathbb{R}$ , starting at zero and killed at the first positive instant when they reach 0 (this instant can possibly be infinite). For  $\xi \in \mathcal{F}$ , let us denote  $\zeta(\xi) := \inf\{s > 0, \xi(s) = 0\}$  for the length of the excursion  $\xi$ . For  $h > 0$ , let  $\mathcal{F}_{h,-}$  and  $\mathcal{F}_{h,+}$  denote respectively the set of excursions which height is strictly less than  $h$  and the set of excursions higher than  $h$  :

$$\mathcal{F}_{h,-} := \left\{ \xi \in \mathcal{F}, \sup_{[0,\zeta]} \xi < h \right\}, \quad \mathcal{F}_{h,+} := \left\{ \xi \in \mathcal{F}, \sup_{[0,\zeta]} \xi \geq h \right\}.$$

Let  $\mathcal{N}$  be the measure defined on  $\mathcal{F}$  as in Subsection 3.5 of [14]. Let  $S^{h,-}$  and  $S^{h,+}$  be two independent pure jumps subordinators with Lévy measures respectively  $\zeta\mathcal{N}(\mathcal{F}_{h,-} \cap \cdot)$  and  $\zeta\mathcal{N}(\mathcal{F}_{h,+} \cap \cdot)$ , the image measures of respectively  $\mathcal{N}(\mathcal{F}_{h,-} \cap \cdot)$  and  $\mathcal{N}(\mathcal{F}_{h,+} \cap \cdot)$  by  $\zeta$ . The sum of these measures equals  $\zeta\mathcal{N}$  so  $S := S^{h,-} + S^{h,+}$  is a pure jumps subordinator with Lévy measure  $\zeta\mathcal{N}$ . Let also  $T_h$  be an exponential random variable with parameter  $\mathcal{N}(\mathcal{F}_{h,+})$ , independent from  $S^{h,-}$  and  $S^{h,+}$ . According to Lemma 3.9 of [14] we have

$$\forall j \geq 1, \tilde{m}_j - \tilde{L}_j^\# \stackrel{\mathcal{L}}{=} S^{h_t,-}(T_{h_t}).$$

Recall that the sequence  $(\tilde{m}_j - \tilde{L}_j^\#)_{j \geq 1}$  is *iid* because of Remark 2.2. Therefore, on an enlarged probability space, there is an *iid* sequence  $(S_j^{h_t,-}, T_{h_t,j})_{j \geq 1}$  such that for all  $j \geq 1$ ,

$$S_j^{h_t,-} \perp\!\!\!\perp T_{h_t,j}, \quad S_j^{h_t,-} \stackrel{\mathfrak{L}}{=} S^{h_t,-}, \quad T_{h_t,j} \stackrel{\mathfrak{L}}{=} T_{h_t} \quad \text{and} \quad S_j^{h_t,-}(T_{h_t,j}) = \tilde{m}_j - \tilde{L}_j^\#. \quad (4.129)$$

Let  $d := \mathcal{N}(\mathcal{F}_{h_t,+})T_{h_t}$  and more generally  $d_j := \mathcal{N}(\mathcal{F}_{h_t,+})T_{h_t,j}$ . Then,  $d_j \stackrel{\mathcal{L}}{\sim} \mathcal{E}(1)$  and the sequence  $(d_j)_{j \geq 1}$  is *iid*. For large  $t$ , it is natural to approximate  $S^{h_t,-}(T_{h_t})$  by a multiple of  $d$ , for this we write

$$S^{h_t,-}(T_{h_t}) - e^{\kappa h_t} d/q = \left( S^{h_t,-}(T_{h_t}) - S(T_{h_t}) \right) + \left( S(T_{h_t}) - \mathbb{E}[S(1)]T_{h_t} \right) + \left( \mathbb{E}[S(1)]T_{h_t} - e^{\kappa h_t} d/q \right). \quad (4.130)$$

For the first term, using the expression of  $S$  in terms of  $S^{h_t,+}$  and  $S^{h_t,-}$ , the independence between  $S^{h_t,+}$  and  $T_{h_t}$ , the definition of  $S^{h_t,+}$ , Cauchy-Schwarz's inequality and the definition of  $S$ , we have

$$\begin{aligned} \mathbb{E} \left[ \left| S^{h_t,-}(T_{h_t}) - S(T_{h_t}) \right| \right] &= \mathbb{E} \left[ S^{h_t,+}(T_{h_t}) \right] = \frac{1}{\mathcal{N}(\mathcal{F}_{h_t,+})} \mathbb{E} \left[ S^{h_t,+}(1) \right] \\ &= \frac{1}{\mathcal{N}(\mathcal{F}_{h_t,+})} \int_{\mathcal{F}_{h_t,+}} \zeta(\xi) \mathcal{N}(d\xi) \leq \frac{1}{\sqrt{\mathcal{N}(\mathcal{F}_{h_t,+})}} \sqrt{\int \zeta^2(\xi) \mathcal{N}(d\xi)}, \\ &= \frac{1}{\sqrt{\mathcal{N}(\mathcal{F}_{h_t,+})}} \sqrt{\text{Var}(S(1))}. \end{aligned} \quad (4.131)$$

$\text{Var}(S(1))$  is indeed finite thanks to Lemma 3.10 of [14]. For the second term in the right hand side of (4.130), using Cauchy-Schwarz's inequality and the independence between  $S$  and

$T_{h_t}$ ,

$$\mathbb{E}[|S(T_{h_t}) - \mathbb{E}[S(1)]T_{h_t}|] \leq \sqrt{\mathbb{E}[(S(T_{h_t}) - \mathbb{E}[S(1)]T_{h_t})^2]} = \frac{1}{\sqrt{\mathcal{N}(\mathcal{F}_{h_t,+})}} \sqrt{\text{Var}(S(1))}. \quad (4.132)$$

For the third term in the right hand side of (4.130), using the definition of  $d$ ,

$$\mathbb{E}[S(1)]T_{h_t} - e^{\kappa h_t}d/q = (\mathbb{E}[S(1)]/\mathcal{N}(\mathcal{F}_{h_t,+}) - e^{\kappa h_t}/q)d.$$

Then, recall from Proposition 3.12 and Lemma 3.11 of [14] that

$$q := \mathcal{N}(\mathcal{F}_{1,+}) \times (e^\kappa - \mathbb{E}[e^{\kappa V_1(\tau(V_1,]-\infty,0])}])/\mathbb{E}[S(1)],$$

and

$$\mathcal{N}(\mathcal{F}_{h_t,+}) = e^{-\kappa h_t} \mathcal{N}(\mathcal{F}_{1,+}) \times \left( e^\kappa - \mathbb{E}[e^{\kappa V_1(\tau(V_1,]-\infty,0])}] \right) + \mathcal{O}_{t \rightarrow +\infty}(e^{-2\kappa h_t}).$$

We thus get

$$\text{a.s. } \mathbb{E}[S(1)]T_{h_t} - e^{\kappa h_t}d/q = d \times \mathcal{O}_{t \rightarrow +\infty}(e^{-\kappa h_t}). \quad (4.133)$$

Let us define  $D^t := e^{\kappa h_t}d/q$  and more generally  $D_j^t := e^{\kappa h_t}d_j/q$ . We then have indeed  $D_j^t \stackrel{\mathcal{L}}{\sim} \mathcal{E}(e^{-\kappa h_t}q)$  and the fact that the sequence sequence  $(D_j^t)_{j \geq 1}$  is *iid.* (4.133) becomes

$$\text{a.s. } |\mathbb{E}[S(1)]T_{h_t} - D^t| \leq D^t \times C e^{-2\kappa h_t}, \quad (4.134)$$

where  $C$  is a positive constant. Using Markov's inequality, (4.131) and (4.132), and the asymptotic of  $\mathcal{N}(\mathcal{F}_{h_t,+})$  given by Lemma 3.11 of [14], we get the existence of a positive constant  $C$  such that for  $t$  large enough,

$$\mathbb{P}\left(|S^{h_t,-}(T_{h_t}) - S(T_{h_t})| + |S(T_{h_t}) - \mathbb{E}[S(1)]T_{h_t}| \geq e^{6\kappa h_t/10}\right) \leq C e^{-\kappa h_t/10}.$$

$$\mathbb{P}\left(D^t \leq e^{9\kappa h_t/10}\right) = \mathbb{P}\left(d \leq q e^{-\kappa h_t/10}\right) \leq q e^{-\kappa h_t/10}.$$

As a consequence there is a positive constant  $C$  such that for  $h_t$  large enough,

$$\mathbb{P}\left(|S^{h_t,-}(T_{h_t}) - S(T_{h_t})| + |S(T_{h_t}) - \mathbb{E}[S(1)]T_{h_t}| \geq e^{-3\kappa h_t/10} D^t\right) \leq C e^{-\kappa h_t/10}.$$

Combining with (4.134) and putting into (4.130) we get

$$\mathbb{P}\left(|S^{h_t,-}(T_{h_t}) - D^t| \geq e^{-2\kappa h_t/10} D^t\right) \leq C e^{-\kappa h_t/10}.$$

Combining with (4.129) we get

$$\forall j \geq 1, \mathbb{P}\left(|\tilde{m}_j - \tilde{L}_j^\# - D_j^t| \geq e^{-2\kappa h_t/10} D_j^t\right) \leq C e^{-\kappa h_t/10}. \quad (4.135)$$

Then, we have  $\mathbb{P}(D_j^t \leq e^{(1-\delta/2)\kappa h_t}) = \mathbb{P}(d_j \leq q e^{-\delta\kappa h_t/2}) \leq q e^{-\delta\kappa h_t/2}$  so putting into (4.127) and (4.128) we get for all  $j \geq 1$

$$\mathbb{P}\left((\tilde{L}_j^\# - \tilde{L}_{j-1}) + (\tilde{\tau}_j(h_t) - \tilde{m}_j) + (\tilde{L}_j - \tilde{\tau}_j(h_t)) > 2e^{-\delta\kappa h_t/2} D_j^t\right) \leq 2e^{-h_t} + q e^{-\delta\kappa h_t/2}. \quad (4.136)$$

Combining (4.135) and (4.136), and putting into (4.126) we get the result for  $t$  large enough.  $\square$

**4.3. Proof of some facts and lemmas.** This subsection is devoted to the justification of the facts stated in Subsection 2.1, which mainly come from [14]. As these results are included in Section 2, we prove them under the hypotheses of Theorems 1.4 and 1.9 :  $0 < \kappa < 1$ ,  $V$  has unbounded variation and there exists  $p > 1$  such that  $V(1) \in L^p$ .

In the facts and lemmas considered here, the value of the constant  $c$  is not important (as long as it is positive) so we allow it to decrease from line to line and it is implicit that the estimates are true for  $t$  large enough.

*Proof.* of Lemma 2.1 (2.23) is only Lemma 4.16 of [14]. For (2.24), note that  $\{N_t \leq \tilde{n}_t\} = \{H(m_{\tilde{n}_t+1}) > t\}$  and  $H(\tilde{m}_{\tilde{n}_t+1}) = H(m_{\tilde{n}_t+1})$  on  $\{V \in \mathcal{V}_t\}$  (since  $\tilde{n}_t < n_t$ ). Let  $t$  be large enough so that  $(1 - 2/\log(h_t))(1 + e^{-\tilde{c}h_t})^{-1} \geq 1/2$ . Using the definitions of  $\mathcal{E}_t^5$  and  $\mathcal{E}_t^7$  (which is possible here since  $\tilde{n}_t < n_t$ ) we get that  $\mathbb{P}(\{N_t \leq \tilde{n}_t\} \cap \{V \in \mathcal{V}_t\} \cap \mathcal{E}_t^5 \cap \mathcal{E}_t^7)$  is less than

$$\begin{aligned} \mathbb{P}(H(\tilde{m}_{\tilde{n}_t+1}) > t, \mathcal{E}_t^5, \mathcal{E}_t^7) &\leq \mathbb{P}\left(\sum_{i=1}^{\tilde{n}_t} e_i S_i^t R_i^t \geq t/2\right) \\ &\leq \mathbb{P}\left(\sup_{1 \leq i \leq \tilde{n}_t} e_i S_i^t R_i^t \geq t/2\tilde{n}_t\right) = 1 - [\mathbb{P}(e_1 S_1^t R_1^t/t < 1/2\tilde{n}_t)]^{\tilde{n}_t} \\ &= 1 - [1 - \mathbb{P}(e_1 S_1^t R_1^t/t \geq 1/2\tilde{n}_t)]^{\tilde{n}_t}. \end{aligned}$$

According to (2.38) applied with some  $\eta \in ]0, (1 - \rho)/3[$  we have that

$$\mathbb{P}(e_1 S_1^t R_1^t/t \geq 1/2\tilde{n}_t) \underset{t \rightarrow +\infty}{\sim} 2^\kappa \mathcal{C}' \mathbb{E}[\mathcal{R}^\kappa] \tilde{n}_t^\kappa e^{-\kappa\phi(t)} = 2^\kappa \mathcal{C}' \mathbb{E}[\mathcal{R}^\kappa] e^{\kappa(\rho-1)\phi(t)}.$$

Since  $\rho < 1$  the later converges to 0 so

$$\begin{aligned} [1 - \mathbb{P}(e_1 S_1^t R_1^t/t \geq 1/2\tilde{n}_t)]^{\tilde{n}_t} &\underset{t \rightarrow +\infty}{\approx} \exp\left(-2^\kappa \mathcal{C}' \mathbb{E}[\mathcal{R}^\kappa] \tilde{n}_t^\kappa e^{\kappa(\rho-1)\phi(t)}\right) \\ &= \exp\left(-2^\kappa \mathcal{C}' \mathbb{E}[\mathcal{R}^\kappa] e^{(\rho+\kappa(\rho-1))\phi(t)}\right). \end{aligned}$$

Let  $\rho' := -(\rho + \kappa(\rho - 1))$ .  $\rho'$  is positive thanks to the hypothesis  $\rho \in ]0, \kappa/(1 + \kappa)[$  so  $e^{(\rho+\kappa(\rho-1))\phi(t)}$  converges to 0 and we deduce that for  $t$  large enough,

$$\mathbb{P}(\{N_t \leq \tilde{n}_t\} \cap \{V \in \mathcal{V}_t\} \cap \mathcal{E}_t^5 \cap \mathcal{E}_t^7) \leq 2\mathcal{C}' \mathbb{E}[\mathcal{R}^\kappa] e^{-\rho'\phi(t)}.$$

Combining with the bounds for  $\mathbb{P}(V \notin \mathcal{V}_t)$  and  $\mathbb{P}(\overline{\mathcal{E}}_t^5) + \mathbb{P}(\overline{\mathcal{E}}_t^7)$  given by respectively Fact 2.3 and (2.25) we get (2.24).  $\square$

Now, let us recall a fact about brownian local time :

**Fact 4.10.** *There is a positive constant  $c$  such that for all  $t$  large enough*

$$\mathbb{P}\left(\sup_{|y| \leq e^{-h_t/3}} |\mathcal{L}_B(\tau(B, 1), y) - \mathcal{L}_B(\tau(B, 1), 0)| > e^{-h_t/9} \mathcal{L}_B(\tau(B, 1), 0)\right) \leq e^{-ch_t}, \quad (4.137)$$

$$\mathbb{P}\left(\sup_{|y| \leq e^{-h_t/8}} |\mathcal{L}_B(\sigma_B(1, 0), y) - 1| \geq e^{-h_t/20}\right) \leq e^{-h_t}, \quad (4.138)$$

$$\mathbb{P}\left(\sup_{y \in \mathbb{R}} \mathcal{L}_B(\sigma_B(1, 0), y) \geq u\right) \leq 2/u, \quad (4.139)$$

*Proof.* (4.137) is (7.11) of [2] applied with  $\delta = e^{-h_t/3}$ ,  $\epsilon = e^{-h_t/9}$ . (4.138) comes from the second Ray-Knight Theorem combined with estimate (7.15) of [2] applied with  $u = e^{-h_t/20}$ ,  $v = e^{-h_t/8}$ . (4.139) comes from the second Ray-Knight Theorem combined with estimate (7.16) of [2].

□

*Proof.* of Fact 2.5

According to the combination of Lemmas 5.20 and 5.21 of [14], there is a positive constant  $c$  such that  $\mathbb{P}(\overline{\mathcal{E}_t^1}) \leq e^{-cht}$ . Since  $h_t$  is ultimately greater than  $\phi(t)$  we get  $\mathbb{P}(\overline{\mathcal{E}_t^1}) \leq e^{-c\phi(t)}$ .

$\mathbb{P}(\overline{\mathcal{E}_t^2}) \leq e^{-c\phi(t)}$  and  $\mathbb{P}(\overline{\mathcal{E}_t^3}) \leq e^{-c\phi(t)}$  come from Fact 4.4 of [14] (respectively the second and third point).

For  $\mathbb{P}(\overline{\mathcal{E}_t^4})$ . Let us fix  $j \geq 1$  and use the Markov property at  $H(\tilde{m}_j)$  and the expression (3.94) of the local time within an hitting time. We get the existence of a Brownian motion  $(B(s), s \geq 0)$ , independent from  $\tilde{V}^{(j)}$ , such that

$$\forall y \in \mathcal{D}_j, \mathcal{L}_X(H(\tilde{L}_j), y) - \mathcal{L}_X(H(\tilde{m}_j), y) = e^{-\tilde{V}^{(j)}(y)} \mathcal{L}_B(\tau(B, A^j(\tilde{L}_j)), A^j(y)),$$

where  $A^j$  is defined in Subsection 2.1. Let  $\tilde{B} := B^j((A^j(\tilde{L}_j))^2)/A^j(\tilde{L}_j)$ . By the scaling property for brownian motion, we have that, conditionally to  $V$ ,  $\tilde{B}$  is a standard Brownian motion. We have

$$\forall y \in \mathcal{D}_j, \mathcal{L}_X(H(\tilde{L}_j), y) - \mathcal{L}_X(H(\tilde{m}_j), y) = e^{-\tilde{V}^{(j)}(y)} A^j(\tilde{L}_j) \mathcal{L}_{\tilde{B}}(\tau(\tilde{B}, 1), A^j(y)/A^j(\tilde{L}_j)).$$

From the definition of  $\mathcal{D}_j$  we have  $\mathcal{D}_j \subset [\tilde{\tau}_j^-(h_t/2), \tilde{\tau}_j^+(h_t/2)]$  so, thanks to (4.124) we have

$$\mathbb{P}\left(\sup_{y \in \mathcal{D}_j} \left|A^j(y)/A^j(\tilde{L}_j)\right| \leq e^{-h_t/3}\right) \geq 1 - e^{-cht}.$$

Combining with estimate (4.137) we get

$$\mathbb{P}\left(\sup_{y \in \mathcal{D}_j} \mathcal{L}_{\tilde{B}}(\tau(\tilde{B}, 1), A^j(y)/A^j(\tilde{L}_j)) > (1 + e^{-h_t/9}) \mathcal{L}_{\tilde{B}}(\tau(\tilde{B}, 1), 0)\right) \leq e^{-cht}$$

so with probability greater than  $1 - e^{-cht}$  we have

$$\begin{aligned} \mathcal{L}_X(H(\tilde{L}_j), \tilde{m}_j) &\leq \sup_{y \in \mathcal{D}_j} \left(\mathcal{L}_X(H(\tilde{L}_j), y) - \mathcal{L}_X(H(\tilde{m}_j), y)\right) \\ &\leq (1 + e^{-h_t/9}) A^j(\tilde{L}_j) \mathcal{L}_{\tilde{B}}(\tau(\tilde{B}, 1), 0) \\ &= (1 + e^{-h_t/9}) \mathcal{L}_X(H(\tilde{L}_j), \tilde{m}_j). \end{aligned}$$

Since  $\tilde{c} > 0$  has been chosen "small enough" in the Introduction, we can assume  $\tilde{c} \leq 1/9$  so  $\mathbb{P}(\overline{\mathcal{E}_t^4}) \leq e^{-cht} \leq e^{-c\phi(t)}$  follows from the above (and the fact that  $h_t$  is ultimately greater than  $\phi(t)$ ).

$\mathbb{P}(\overline{\mathcal{E}_t^5}) \leq e^{-c\phi(t)}$  comes from Fact 4.3 of [14].

$\tilde{c} > 0$  has been fixed "small enough" in the Introduction. We can assume that it was chosen so small such that Proposition 4.5 of [14] apply with  $\tilde{c}$  instead the constant  $\epsilon/7$  there.  $\mathbb{P}(\overline{\mathcal{E}_t^6}) \leq e^{-cht} \leq e^{-c\phi(t)}$  and  $\mathbb{P}(\overline{\mathcal{E}_t^7}) \leq e^{-cht} \leq e^{-c\phi(t)}$  thus come from Proposition 4.5 of [14] (and the fact that  $h_t$  is ultimately greater than  $\phi(t)$ ).

The second point of the fact is Lemma 4.11 of [14].

□

*Proof.* of Lemma 2.6



According to (2.39) and Chernoff inequality we have for any  $j \geq 1$

$$P\left(R_j^t \geq e^{h_t/8}\right) \leq e^{-\lambda_0 e^{h_t/8}/2}. \quad (4.140)$$

We have

$$\begin{aligned} P\left(R_j^t \leq e^{-\epsilon h_t/4}\right) &\leq P\left(R_j^t \leq e^{-h_t^{1/3}}\right) \leq 2P\left(I(V^\uparrow) \leq 2e^{-h_t^{1/3}}\right) + 2e^{-\delta \kappa h_t/3} \\ &\leq 2e^{-K_0 e^{h_t^{1/3}}/2} + 2e^{-\delta \kappa h_t/3}, \end{aligned} \quad (4.141)$$

where, for the last two inequalities, we used Lemma 2.13 (applied with  $z_t = e^{h_t^{1/3}}$ ,  $a = 2$ ) and (1.7), and where  $K_0$  is the constant in (1.7). The term  $2e^{-\delta \kappa h_t/3}$  is only necessary in the case  $V = W_\kappa$  (because of (2.50)). Combining (4.140) and (4.141) we get that (2.28) is satisfied with probability at least  $1 - e^{-\epsilon h_t}$ .

According to Fact 4.8 we have that (2.29), (2.30), (2.31) and (2.32) are satisfied with probability at least  $1 - e^{-\epsilon h_t}$ .

According to (2.25) we have  $\mathbb{P}(\cup_{i=1}^7 \overline{\mathcal{E}}_t^i) \leq e^{-L\phi(t)}$ . Then,

$$e^{-L\phi(t)/2} P\left(P^V(\cup_{i=1}^7 \overline{\mathcal{E}}_t^i) > e^{-L\phi(t)/2}\right) \leq E\left[P^V(\cup_{i=1}^7 \overline{\mathcal{E}}_t^i)\right] = \mathbb{P}\left(\cup_{i=1}^7 \overline{\mathcal{E}}_t^i\right) \leq e^{-L\phi(t)},$$

We thus deduce that (2.33) is satisfied with probability at least  $1 - e^{-L\phi(t)/2}$ . Combing all this with Fact 2.3, we get Lemma 2.6 for a suitably chosen constant  $c$ .  $\square$

*Proof.* of Fact 2.7

The ideas are very similar to the ones used for the proof of Lemma 5.3 of [2]. Since we are in the more general context of a Lévy potential and since we do not prove exactly the same thing (here, we work conditionally to the environment), we give the details.

Recall that  $\epsilon > 0$  has been fixed in the definition of  $\mathcal{G}_t$ .  $\tilde{c} > 0$  has been fixed "small enough" in the Introduction (and we already fixed some constraints about how small it must be in the proof of Fact 2.5). We can assume further that it was chosen so small such that  $\tilde{c} < \min(1/20, \epsilon/4)$ . Let us fix an environment  $v \in \mathcal{G}_t$ ,  $z \in [0, 1 - 4/\log(h_t)]$  and  $k \leq n_t$ . We put  $\tilde{x}_t := (1 + e^{-\tilde{c}h_t})^{-1}x_t$ ,  $Z := (1 - z)/(1 - e^{-\tilde{c}h_t})R_k^t$ . To study  $\mathcal{E}_t^8(v, k, z)$ , we look at its intersections with the events  $\{\sigma_{X_{\tilde{m}_k}}(t\tilde{x}_t, \tilde{m}_k) < H_{X_{\tilde{m}_k}}(\tilde{L}_k)\}$  and  $\{\sigma_{X_{\tilde{m}_k}}(t\tilde{x}_t, \tilde{m}_k) > H_{X_{\tilde{m}_k}}(\tilde{L}_k)\}$ . We have

$$\begin{aligned} &\mathcal{E}_t^8(v, k, z) \cap \{\sigma_{X_{\tilde{m}_k}}(t\tilde{x}_t, \tilde{m}_k) < H_{X_{\tilde{m}_k}}(\tilde{L}_k)\} \\ &= \left\{ tZ \leq t\tilde{x}_t, \sup_{\mathcal{D}_k} \mathcal{L}_{X_{\tilde{m}_k}}(t(1-z), \cdot) \geq tx_t, \sigma_{X_{\tilde{m}_k}}(t\tilde{x}_t, \tilde{m}_k) < H_{X_{\tilde{m}_k}}(\tilde{L}_k) < H_{X_{\tilde{m}_k}}(\tilde{L}_{k-1}) \right\}. \end{aligned} \quad (4.142)$$

On the above event,  $\sigma_{X_{\tilde{m}_k}}(tZ, \tilde{m}_k)$  is finite and the diffusion stays in  $[\tilde{L}_{k-1}, \tilde{L}_k]$  until this time.  $\sigma_{X_{\tilde{m}_k}}(tZ, \tilde{m}_k)$  is thus equal to

$$I := \int_{\tilde{L}_{k-1}}^{\tilde{L}_k} e^{-\tilde{v}^{(k)}(x)} \mathcal{L}_B(\sigma_B(tZ, 0), A^k(x)) dx, \quad (4.143)$$

where  $B$  is the brownian motion driving the diffusion  $X_{\tilde{m}_k}$  and  $A^k$  is defined in Subsection 2.1. Let  $\tilde{B} := B((tZ)^2 \cdot)/tZ$  and  $\tilde{A}^k(\cdot) := A^k(\cdot)/tZ$ . From the scaling property  $\tilde{B}$  is still a brownian motion. We can write

$$I = tZ \int_{\tilde{L}_{k-1}}^{\tilde{L}_k} e^{-\tilde{v}^{(k)}(x)} \mathcal{L}_{\tilde{B}}(\sigma_{\tilde{B}}(1, 0), \tilde{A}^k(x)) dx. \quad (4.144)$$

Since  $v \in \mathcal{G}_t$  it satisfies (2.28). Recall also that  $t \geq e^{h_t}$ . For  $t$  large enough so that  $4e^{-h_t/8}/(1 - e^{-\tilde{c}h_t}) \log(h_t) \geq e^{-h_t/4}$  we thus have  $tZ \geq e^{3h_t/4}$  for any choice of  $z \in [0, 1 - 4/\log(h_t)]$ . Combining this with the fact that  $v$  satisfies (2.29) and (2.30) we get

$$\sup_{[\tilde{\tau}_k^-(h_t/2), \tilde{\tau}_k^+(h_t/2)]} |\tilde{A}^k(\cdot)| \leq \left( |A^k(\tilde{\tau}_k^-(h_t/2))| \vee A^k(\tilde{\tau}_k^+(h_t/2)) \right) / tZ \leq e^{-h_t/8}.$$

Combining with (4.138) we get

$$P^v \left( \sup_{x \in [\tilde{\tau}_k^-(h_t/2), \tilde{\tau}_k^+(h_t/2)]} |\mathcal{L}_{\tilde{B}}(\sigma_{\tilde{B}}(1, 0), \tilde{A}^k(x)) - 1| \geq e^{-h_t/20} \right) \leq e^{-h_t}, \quad (4.145)$$

and we deduce that

$$P^v \left( \left| \int_{\tilde{\tau}_k^-(h_t/2)}^{\tilde{\tau}_k^+(h_t/2)} e^{-\tilde{v}^{(k)}(x)} \mathcal{L}_{\tilde{B}}(\sigma_{\tilde{B}}(1, 0), \tilde{A}^k(x)) dx - R_k^t \right| \leq e^{-h_t/20} R_k^t \right) \geq 1 - e^{-h_t}. \quad (4.146)$$

Since  $v \in \mathcal{G}_t$  it satisfies (2.31) and (2.32). Combining with (4.139) (applied with  $u = e^{ch_t/2}$ ) and the lower bound for  $R_k^t$  in (2.28) we get

$$P^v \left( \int_{\tilde{\tau}_k^-(h_t/2)}^{\tilde{\tau}_k^+(h_t/2)} e^{-\tilde{v}^{(k)}(x)} \mathcal{L}_{\tilde{B}}(\sigma_{\tilde{B}}(1, 0), A^k(x)) dx \leq e^{-ch_t/2} \leq e^{-ch_t/4} R_k^t \right) \geq 1 - e^{-ch_t}, \quad (4.147)$$

$$P^v \left( \int_{\tilde{\tau}_k^-(h_t/2)}^{\tilde{\tau}_k^+(h_t/2)} e^{-\tilde{v}^{(k)}(x)} \mathcal{L}_{\tilde{B}}(\sigma_{\tilde{B}}(1, 0), A^k(x)) dx \leq e^{-ch_t/2} \leq e^{-ch_t/4} R_k^t \right) \geq 1 - e^{-ch_t}. \quad (4.148)$$

Now putting (4.146), (4.147) and (4.148) into (4.144) (and combining with the fact that  $\tilde{c} < \min(1/20, \epsilon/4)$ ) we get

$$P^v \left( \mathcal{E}_{neg}^1 := \left\{ |I - tZR_k^t| > e^{-\tilde{c}h_t} tZR_k^t \right\} \right) \leq e^{-ch_t}.$$

Combining with the definition of  $Z$  we see that  $t(1 - z) \leq I$  on the complementary of  $\mathcal{E}_{neg}^2$ . Combining this with (4.142) we get that  $\mathcal{E}_t^8(v, k, z) \cap \{\sigma_{X_{\tilde{m}_k}}(t\tilde{x}_t, \tilde{m}_k) < H_{X_{\tilde{m}_k}}(\tilde{L}_k)\}$  is included into

$$\left\{ tZ \leq t\tilde{x}_t, \sup_{\mathcal{D}_k} \mathcal{L}_{X_{\tilde{m}_k}}(\sigma_{X_{\tilde{m}_k}}(tZ, \tilde{m}_k), \cdot) \geq tx_t, \sigma_{X_{\tilde{m}_k}}(t\tilde{x}_t, \tilde{m}_k) < H_{X_{\tilde{m}_k}}(\tilde{L}_k) < H_{X_{\tilde{m}_k}}(\tilde{L}_{k-1}) \right\} \cup \mathcal{E}_{neg}^1. \quad (4.149)$$

On the main event,  $\sigma_{X_{\tilde{m}_k}}(tZ, \tilde{m}_k)$  is finite. On this event,  $\mathcal{L}_{X_{\tilde{m}_k}}(\sigma_{X_{\tilde{m}_k}}(tZ, \tilde{m}_k), y)$  is thus equal to

$$L(y) := e^{-\tilde{v}^{(k)}(y)} \mathcal{L}_B(\sigma_B(tZ, 0), A^k(y)).$$

Here again, for  $\tilde{B} := B((tZ)^2)/tZ$  and  $\tilde{A}^k(\cdot) := A^k(\cdot)/tZ$ . We have

$$L(y) = tZe^{-\tilde{v}^{(k)}(y)} \mathcal{L}_{\tilde{B}}(\sigma_{\tilde{B}}(1, 0), \tilde{A}^k(y)).$$

Since  $\mathcal{D}_k \subset [\tilde{\tau}_k^-(h_t/2), \tilde{\tau}_k^+(h_t/2)]$  we can apply (4.145) (together with  $\tilde{c} < 1/20$ ) and get

$$P^v \left( \mathcal{E}_{neg}^2 := \left\{ \exists y \in \mathcal{D}_k, L(y) \geq (1 + e^{-\tilde{c}h_t})tZ \right\} \right) \leq e^{-ch_t}. \quad (4.150)$$

On the big event in (4.149) we thus have both  $(1 + e^{-\tilde{c}h_t})tZ > tx_t$  and  $tZ \leq t\tilde{x}_t$ , (expect possibly on the event  $\mathcal{E}_{neg}^2$ ). Since these two inequalities are not compatible we get

$$P^v \left( \mathcal{E}_t^8(v, k, z) \cap \{\sigma_{X_{\tilde{m}_k}}(t\tilde{x}_t, \tilde{m}_k) < H_{X_{\tilde{m}_k}}(\tilde{L}_k)\} \right) \leq P^v(\mathcal{E}_{neg}^1) + P^v(\mathcal{E}_{neg}^2) \leq 2e^{-ch_t}. \quad (4.151)$$

We now study the case where  $\sigma_{X_{\tilde{m}_k}}(t\tilde{x}_t, \tilde{m}_k) > H_{X_{\tilde{m}_k}}(\tilde{L}_k)$ . First, we have

$$\begin{aligned} & \mathcal{E}_t^8(v, k, z) \cap \{\sigma_{X_{\tilde{m}_k}}(t\tilde{x}_t, \tilde{m}_k) > H_{X_{\tilde{m}_k}}(\tilde{L}_k)\} \\ &= \left\{ tZ \leq t\tilde{x}_t, \sup_{\mathcal{D}_k} \mathcal{L}_{X_{\tilde{m}_k}}(t(1-z), \cdot) \geq tx_t, H_{X_{\tilde{m}_k}}(\tilde{L}_k) < H_{X_{\tilde{m}_k}}(\tilde{L}_{k-1}) \wedge \sigma_{X_{\tilde{m}_k}}(t\tilde{x}_t, \tilde{m}_k) \right\} \\ &\subset \left\{ \sup_{\mathcal{D}_k} \mathcal{L}_{X_{\tilde{m}_k}}(t(1-z), \cdot) \geq tx_t, \mathcal{L}_{X_{\tilde{m}_k}}(H_{X_{\tilde{m}_k}}(\tilde{L}_k), \tilde{m}_k) < t\tilde{x}_t \right\} \\ &\subset \left\{ \sup_{\mathcal{D}_k} \mathcal{L}_{X_{\tilde{m}_k}}(H_{X_{\tilde{m}_k}}(\tilde{L}_k), \cdot) \geq tx_t, \mathcal{L}_{X_{\tilde{m}_k}}(H_{X_{\tilde{m}_k}}(\tilde{L}_k), \tilde{m}_k) < t\tilde{x}_t \right\} \cup \overline{\mathcal{E}_t^1}. \end{aligned}$$

Indeed,  $\sup_{\mathcal{D}_k} \mathcal{L}_{X_{\tilde{m}_k}}(t(1-z), \cdot) \leq \sup_{\mathcal{D}_k} \mathcal{L}_{X_{\tilde{m}_k}}(H_{X_{\tilde{m}_k}}(\tilde{L}_k), \cdot)$  on  $\{t(1-z) \leq H_{X_{\tilde{m}_k}}(\tilde{L}_k)\}$  and  $\sup_{\mathcal{D}_k} \mathcal{L}_{X_{\tilde{m}_k}}(t(1-z), \cdot) = \sup_{\mathcal{D}_k} \mathcal{L}_{X_{\tilde{m}_k}}(H_{X_{\tilde{m}_k}}(\tilde{L}_k), \cdot)$  on  $\{H_{X_{\tilde{m}_k}}(\tilde{L}_k) < t(1-z)\} \cap \mathcal{E}_t^1$ . Since we are dealing with  $X_{\tilde{m}_k}$ , the diffusion shifted at time  $H(\tilde{m}_k)$ , we can see that the main event above is included in  $\overline{\mathcal{E}_t^4}$ . We thus get

$$\mathcal{E}_t^8(v, k, z) \cap \{\sigma_{X_{\tilde{m}_k}}(t\tilde{x}_t, \tilde{m}_k) > H_{X_{\tilde{m}_k}}(\tilde{L}_k)\} \subset \overline{\mathcal{E}_t^1} \cup \overline{\mathcal{E}_t^4}. \quad (4.152)$$

(2.35) follows easily from the combination of (4.151) and (4.152) applied with  $z = 1 - 4/\log(h_t)$ . Then, the right hand sides of (4.151) and (4.152) do not depend on  $z$  (which is arbitrary in  $[0, 1 - 4/\log(h_t)]$ ) and  $H(\tilde{m}_k)/t$  is independent from  $X_{\tilde{m}_k}$ . We can thus replace  $z$  by  $H(\tilde{m}_k)/t$  in (4.151) and (4.152) (at least on  $\{N_t \geq k, H(\tilde{m}_k)/t \leq 1 - 4/\log(h_t)\}$ ). Using the combination of (4.151) and (4.152) to study the union of events in (2.34) we get

$$P^v \left( \bigcup_{k=1}^{n_t} \{N_t \geq k, H(\tilde{m}_k)/t \leq 1 - 4/\log(h_t)\} \cap \mathcal{E}_t^8(v, k, H(\tilde{m}_k)/t) \right) \leq 2n_t e^{-ch_t} + P^v \left( \overline{\mathcal{E}_t^1} \cup \overline{\mathcal{E}_t^4} \right).$$

Since  $v \in \mathcal{G}_t$  it satisfies (2.33). We thus have  $P^v(\overline{\mathcal{E}_t^1} \cup \overline{\mathcal{E}_t^4}) \leq e^{-L\phi(t)/2}$  (where  $L$  is the constant defined in Fact 2.5) and from the definition of  $h_t$  and  $\phi(t)$  we have easily  $2n_t e^{-ch_t} \leq e^{-c\phi(t)}$  for large  $t$ . For  $t$  large enough we thus get (2.34).

We now prove (2.36). Let us fix an environment  $v \in \mathcal{G}_t$ ,  $z \in [0, 1]$  and  $k \leq n_t$ . Inversing the local time in the definition of  $\mathcal{E}_t^9(v, k, z)$  we get that  $\mathcal{E}_t^9(v, k, z)$  coincides with the event

$$\left\{ R_k^t/x_t < (1 - e^{-\tilde{c}h_t})(1 - z), \sigma_{X_{\tilde{m}_k}}(t/x_t, \tilde{m}_k) \wedge H_{X_{\tilde{m}_k}}(\tilde{L}_k) \geq t(1 - z), \right. \\ \left. H_{X_{\tilde{m}_k}}(\tilde{L}_k) < H_{X_{\tilde{m}_k}}(\tilde{L}_{k-1}) \right\}. \quad (4.153)$$

We have to distinguish the cases  $H_{X_{\tilde{m}_k}}(\tilde{L}_k) > \sigma_{X_{\tilde{m}_k}}(t/x_t, \tilde{m}_k)$  and  $\sigma_{X_{\tilde{m}_k}}(t/x_t, \tilde{m}_k) > H_{X_{\tilde{m}_k}}(\tilde{L}_k)$ . On  $\mathcal{E}_t^9(v, k, z) \cap \{\sigma_{X_{\tilde{m}_k}}(t\tilde{x}_t, \tilde{m}_k) < H_{X_{\tilde{m}_k}}(\tilde{L}_k)\}$ ,  $\sigma_{X_{\tilde{m}_k}}(t/x_t, \tilde{m}_k)$  is finite and the diffusion stays in  $[\tilde{L}_{k-1}, \tilde{L}_k]$  until this time. On this event,  $\sigma_{X_{\tilde{m}_k}}(t/x_t, \tilde{m}_k)$  is thus equal to

$$I' := \int_{\tilde{L}_{k-1}}^{\tilde{L}_k} e^{-\tilde{v}^{(k)}(x)} \mathcal{L}_B(\sigma_B(t/x_t, 0), A^k(x)) dx,$$

where, as in (4.143),  $B$  is the brownian motion driving the diffusion  $X_{\tilde{m}_k}$ . Now, let  $\tilde{B} := B((t/x_t)^2 \cdot)/(t/x_t)$  and  $\tilde{A}^k(\cdot) := A^k(\cdot)/(t/x_t)$ . From the scaling property  $\tilde{B}$  is still a brownian motion. We can write

$$I' = (t/x_t) \int_{\tilde{L}_{k-1}}^{\tilde{L}_k} e^{-\tilde{v}^{(k)}(x)} \mathcal{L}_{\tilde{B}}(\sigma_{\tilde{B}}(1, 0), \tilde{A}^k(x)) dx.$$

From  $t \geq e^{h_t}$ , the definition (2.27) of  $x_t$  and the definition (2.22) of  $h_t$  we see that  $t/x_t \geq e^{3h_t/4}$  for  $t$  large enough. Recall also that  $v \in \mathcal{G}_t$  implies that  $v$  satisfies (2.29) and (2.30). We thus get

$$\sup_{[\tilde{\tau}_k^-(h_t/2), \tilde{\tau}_k^+(h_t/2)]} |\tilde{A}^k(\cdot)| \leq \left( |A^k(\tilde{\tau}_k^-(h_t/2))| \vee A^k(\tilde{\tau}_k^+(h_t/2)) \right) / (t/x_t) \leq e^{-h_t/8},$$

We can now proceed as in the proof of (2.34) to get analogues of (4.146), (4.147) and (4.148). We deduce that

$$P^v \left( \mathcal{E}_{neg}^3 := \left\{ |I' - (t/x_t)R_k^t| > e^{-\tilde{c}h_t}(t/x_t)R_k^t \right\} \right) \leq e^{-ch_t}.$$

Now, note from (4.153) that  $\mathcal{E}_t^9(v, k, z) \subset \{(1 - e^{-\tilde{c}h_t})\sigma_{X_{\tilde{m}_k}}(t/x_t, \tilde{m}_k) \geq tR_k^t/x_t\}$ . Since  $\sigma_{X_{\tilde{m}_k}}(t/x_t, \tilde{m}_k) = I'$  on  $\mathcal{E}_t^9(v, k, z) \cap \{\sigma_{X_{\tilde{m}_k}}(t/x_t, \tilde{m}_k) < H_{X_{\tilde{m}_k}}(\tilde{L}_k)\}$  and  $(1 + e^{-\tilde{c}h_t})(1 - e^{-\tilde{c}h_t}) < 1$  we deduce that  $\mathcal{E}_t^9(v, k, z) \cap \{\sigma_{X_{\tilde{m}_k}}(t/x_t, \tilde{m}_k) < H_{X_{\tilde{m}_k}}(\tilde{L}_k)\} \subset \mathcal{E}_{neg}^3$ . Then,

$$P^v \left( \mathcal{E}_t^9(v, k, z) \cap \{\sigma_{X_{\tilde{m}_k}}(t/x_t, \tilde{m}_k) < H_{X_{\tilde{m}_k}}(\tilde{L}_k)\} \right) \leq e^{-ch_t}. \quad (4.154)$$

We now study the case where  $\sigma_{X_{\tilde{m}_k}}(t/x_t, \tilde{m}_k) > H_{X_{\tilde{m}_k}}(\tilde{L}_k)$ , following the ideas of the proof of Lemma 5.3 of [2]. Recall that  $B$  is the brownian motion driving the diffusion  $X_{\tilde{m}_k}$ . We have

$$\sigma_{X_{\tilde{m}_k}}(t/x_t, \tilde{m}_k) > H_{X_{\tilde{m}_k}}(\tilde{L}_k) \Leftrightarrow \sigma_B(t/x_t, 0) > \tau(B, A^k(\tilde{L}_k)) \Leftrightarrow t/x_t > \mathcal{L}_B \left( \tau(B, A^k(\tilde{L}_k)), 0 \right)$$

Now, let  $\tilde{B} := B((A^k(\tilde{L}_k))^2)/A^k(\tilde{L}_k)$ . From the scaling property  $\tilde{B}$  is still a brownian motion.  $\mathcal{L}_B \left( \tau(B, A^k(\tilde{L}_k)), 0 \right) = A^k(\tilde{L}_k) \mathcal{L}_{\tilde{B}} \left( \tau(\tilde{B}, 1), 0 \right)$ . Note that from the definition of  $e_k$  given in Subsection 2.1 we have  $\mathcal{L}_{\tilde{B}} \left( \tau(\tilde{B}, 1), 0 \right) = e_k$ . As a consequence,

$$\sigma_{X_{\tilde{m}_k}}(t/x_t, \tilde{m}_k) > H_{X_{\tilde{m}_k}}(\tilde{L}_k) \Leftrightarrow t/x_t > A^k(\tilde{L}_k)e_k \Leftrightarrow tR_k^t/x_t > A^k(\tilde{L}_k)e_k R_k^t.$$

According to the trivial inequality

$$A^k(\tilde{L}_j) = \int_{\tilde{m}_j}^{\tilde{L}_j} e^{-(v(u)-v(\tilde{m}_j))} du \geq \int_{\tilde{\tau}_j^+(h_t/2)}^{\tilde{L}_j} e^{v(u)-v(\tilde{m}_j)} du = S_j^t$$

and the definition of  $\mathcal{E}_t^7$  we deduce that

$$\left\{ \sigma_{X_{\tilde{m}_k}}(t/x_t, \tilde{m}_k) > H_{X_{\tilde{m}_k}}(\tilde{L}_k) \right\} \subset \left\{ tR_k^t/x_t > (1 + e^{-\tilde{c}h_t})^{-1} H_{X_{\tilde{m}_k}}(\tilde{L}_k) \right\} \cup \overline{\mathcal{E}_t^7}$$

Now, note from (4.153) that  $\mathcal{E}_t^9(v, k, z) \subset \{(1 - e^{-\tilde{c}h_t})H_{X_{\tilde{m}_k}}(\tilde{L}_k) \geq tR_k^t/x_t\}$ . Since  $(1 + e^{-\tilde{c}h_t})(1 - e^{-\tilde{c}h_t}) < 1$ , the inequality  $tR_k^t/x_t > (1 + e^{-\tilde{c}h_t})^{-1} H_{X_{\tilde{m}_k}}(\tilde{L}_k)$  is in contradiction with the event  $\mathcal{E}_t^9(v, k, z)$ . We thus get

$$\mathcal{E}_t^9(v, k, z) \cap \{\sigma_{X_{\tilde{m}_k}}(t/x_t, \tilde{m}_k) > H_{X_{\tilde{m}_k}}(\tilde{L}_k)\} \subset \overline{\mathcal{E}_t^7} \quad (4.155)$$

Since the right hand sides of (4.154) and (4.155) do not depend on  $z$  (which is arbitrary in  $[0, 1]$ ) and  $H(\tilde{m}_k)/t$  is independent from  $X_{\tilde{m}_k}$ , we can replace  $z$  by  $H(\tilde{m}_k)/t$  in (4.154) and (4.155) (at least on  $\{N_t \geq k\}$ ). We can thus use the combination of (4.154) and (4.155) to study the union of events in (2.36). We get

$$P^v \left( \cup_{k=1}^{n_t} \{N_t \geq k\} \cap \mathcal{E}_t^9(v, k, H(\tilde{m}_k)/t) \right) \leq n_t e^{-ch_t} + P^v \left( \overline{\mathcal{E}_t^7} \right)$$

Since  $v \in \mathcal{G}_t$  it satisfies (2.33). We thus have  $P^v(\overline{\mathcal{E}_t^7}) \leq e^{-L\phi(t)/2}$  (where  $L$  is the constant defined in Fact 2.5) and from the definition of  $h_t$  and  $\phi(t)$  we have easily  $n_t e^{-ch_t} \leq e^{-c\phi(t)}$  for large  $t$ . For  $t$  large enough we thus get (2.36).  $\square$

*Proof.* of Fact 2.8 (2.37) and (2.38) are included into Lemma 4.15 of [14] while (2.39) comes from Proposition 3.8 of [14] applied with  $h = h_t$ .  $\square$

*Proof.* of Lemma 2.9

First, note that from (2.22) and the definition of  $n_t$  just after, we have  $2t^\kappa e^{\kappa\delta(\log(\log(t)))^\omega}/q = 2n_t e^{\kappa h_t}/q$ . From the definition of  $N_t$ , we know that on  $\{V \in \mathcal{V}_t\}$ , at time  $t$ ,  $\tilde{m}_{N_t+1}$  has never been reached by the diffusion and neither  $\tilde{L}_{N_t+1}$  (because  $\tilde{L}_{N_t+1} > \tilde{m}_{N_t+1}$ ), we thus have

$$\begin{aligned} \mathbb{P}\left(\sup_{[0,t]} X \geq 2n_t e^{\kappa h_t}/q\right) &\leq \mathbb{P}\left(\tilde{L}_{N_t+1} \geq 2n_t e^{\kappa h_t}/q\right) + \mathbb{P}(V \notin \mathcal{V}_t) \\ &\leq \mathbb{P}\left(\tilde{L}_{n_t} \geq 2n_t e^{\kappa h_t}/q\right) + \mathbb{P}(V \notin \mathcal{V}_t) + \mathbb{P}(N_t \geq n_t) \\ &\leq \mathbb{P}\left((1 + e^{-c_1 h_t}) \sum_{j=1}^{n_t} D_j^t \geq 2n_t e^{\kappa h_t}/q\right) + e^{-ch_t}, \end{aligned}$$

where  $c$  is a positive constant,  $c_1$  has the same meaning as in Proposition 4.9 and  $t$  is large enough. For the last inequality, we used Proposition 4.9, Fact 2.3 and (2.23).

$$\leq e^{-(1+e^{-c_1 h_t})^{-1}n_t} \left(\mathbb{E}\left[e^{qe^{-\kappa h_t} D_j^t/2}\right]\right)^{n_t} + e^{-ch_t} = e^{(\log(2)-(1+e^{-c_1 h_t})^{-1})n_t} + e^{-ch_t},$$

where we used Markov's inequality, the fact that the sequence  $(qe^{-\kappa h_t} D_j^t)$  is an *iid* sequence of exponential random variable with parameter 1 and the expression of the Laplace transform for the exponential distribution. Since  $\log(2) < 1$  and  $n_t = e^{\kappa(1+\delta)(\log(\log(t)))^\omega} \gg h_t$ , we get the first point for  $t$  large enough.

We now prove the second point. We first note that from (2.22) and the definition of  $\tilde{n}_t$  just after, we have  $t^\kappa e^{(\rho-\kappa)(\log(\log(t)))^\omega}/2q = \tilde{n}_t e^{\kappa h_t}/2q$ . On  $\{V \in \mathcal{V}_t\} \cap \{N_t < n_t\} \cap \mathcal{E}_t^1$  we have  $X(t) \geq \tilde{L}_{N_t-1}$  so

$$\begin{aligned} \mathbb{P}\left(X(t) \leq \tilde{n}_t e^{\kappa h_t}/2q\right) &\leq \mathbb{P}\left(\tilde{L}_{N_t-1} \leq \tilde{n}_t e^{\kappa h_t}/2q\right) + \mathbb{P}(V \notin \mathcal{V}_t) + \mathbb{P}(N_t \geq n_t) + \mathbb{P}\left(\overline{\mathcal{E}_t^1}\right) \\ &\leq \mathbb{P}\left(\tilde{L}_{\tilde{n}_t} \leq \tilde{n}_t e^{\kappa h_t}/2q\right) + \mathbb{P}(V \notin \mathcal{V}_t) + \mathbb{P}(N_t \leq \tilde{n}_t) + \mathbb{P}(N_t \geq n_t) + \mathbb{P}\left(\overline{\mathcal{E}_t^1}\right) \\ &\leq \mathbb{P}\left((1 - e^{-c_1 h_t}) \sum_{j=1}^{\tilde{n}_t} D_j^t \leq \tilde{n}_t e^{\kappa h_t}/2q\right) + e^{-c\phi(t)} \end{aligned}$$

where  $c$  is a positive constant,  $c_1$  has the same meaning as in Proposition 4.9 and  $t$  is large enough. For the last inequality we used Proposition 4.9, Fact 2.3, Lemma 2.1, (2.25) and the fact that  $e^{-ch_t} \leq e^{-c\phi(t)}$  for large  $t$ .

$$\leq e^{(1-e^{-c_1 h_t})^{-1}\tilde{n}_t} \left(\mathbb{E}\left[e^{-2qe^{-\kappa h_t} D_j^t}\right]\right)^{\tilde{n}_t} + e^{-c\phi(t)} = e^{(-\log(3)+(1-e^{-c_1 h_t})^{-1})\tilde{n}_t} + e^{-c\phi(t)},$$

where we used Markov's inequality, the fact that the sequence  $(qe^{-\kappa h_t} D_j^t)_{j \geq 1}$  is an *iid* sequence of exponential random variable with parameter 1 and the expression for the Laplace transform of the exponential distribution. Since  $\log(3) > 1$  and  $\tilde{n}_t = e^{\rho(\log(\log(t)))^\omega} \gg h_t$ , we get the second point for  $t$  large enough.

We now prove (2.42). Note that for a fixed environment  $v \in \mathcal{V}$  we have

$$P^v\left(\inf_{[0,+\infty[} X \leq -r\right) = P^v(H(-r) < H(+\infty)) = \frac{\int_0^{+\infty} e^{v(x)} dx}{\int_{-r}^{+\infty} e^{v(x)} dx} \leq \frac{\int_0^{+\infty} e^{v(x)} dx}{\int_{-r}^{-r/2} e^{v(x)} dx} \quad (4.156)$$

and note that

$$\int_{-r}^{-r/2} e^{V(x)} dx = \int_{r/2}^r e^{V(-x)} dx \stackrel{\mathcal{L}}{=} \int_{r/2}^r e^{-V(x)} dx,$$

where the equality in law comes from the time-reverse property. Applying Lemma 4.2 with  $t = r/2$  and  $a = \sqrt{r}$  we get

$$P \left( \int_{r/2}^r e^{-V(x)} dx \leq \frac{r}{2} e^{\sqrt{r}} \right) \leq e^{c_1 \sqrt{r} - c_2 r/2} + e^{-\kappa \sqrt{r}}, \quad (4.157)$$

where  $c_1$  and  $c_2$  are the constants in the lemma. Then, applying Lemma 4.3 we get for  $r$  large enough,

$$P \left( \int_0^{+\infty} e^{V(x)} dx \geq e^{\sqrt{r}} \right) \leq e^{-\kappa \sqrt{r}/2}. \quad (4.158)$$

Putting (4.157) and (4.158) into (4.156) we get that with  $P$ -probability greater than  $1 - (e^{c_1 \sqrt{r} - c_2 r/2} + e^{-\kappa \sqrt{r}} + e^{-\kappa \sqrt{r}/2})$  :  $P^V \left( \inf_{[0, +\infty[} X \leq -r \right) \leq 2r^{-1}$  (and it is bounded by 1 when this estimates fails) so integrating on  $\mathcal{V}$  with respect to  $P$  we get

$$\mathbb{P} \left( \inf_{[0, +\infty[} X \leq -r \right) \leq 2r^{-1} + e^{c_1 \sqrt{r} - c_2 r/2} + e^{-\kappa \sqrt{r}} + e^{-\kappa \sqrt{r}/2}.$$

(2.42) follows for  $r$  large enough. Finally, (2.43) is included in Lemma 5.18 of [14].

□

**4.4. Almost sure constantness of limsup and liminf.** We now use a classical argument involving Kolmogorov 0 – 1 law to justify the almost sure constantness of the limsup stated in Remark 1.3. We here treat the case of the limsup with the renormalisation  $t \log(\log(t))$ . The same argument can be used with the liminf instead of the limsup, or with any of the other renormalisations used in the paper.

We first fix  $v \in \mathcal{V}$ , a realization of the environment. For any  $n \in \mathbb{N}$ , the process  $X^n := X(\tau(X, n) + \cdot)$  is, according to the Markov property, a diffusion in the environment  $v(n + \cdot)$  and it is independent from  $(X(s), 0 \leq s \leq \tau(X, n))$ . We have

$$\limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t \log(\log(t))} = \limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_{X^n}^*(t)}{(\tau(X, n) + t) \log(\log(\tau(X, n) + t))} = \limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_{X^n}^*(t)}{t \log(\log(t))} \quad (4.159)$$

where the first equality comes from the fact that  $\mathbb{P}$ -almost surely the favorite site  $F^*(t)$  goes to  $+\infty$ . Indeed, the diffusion  $\mathbb{P}$ -almost surely converges to  $+\infty$  and as we can see from the results of Subsection 1.1,  $\mathcal{L}_X^*(t)$  converges  $\mathbb{P}$ -almost surely to infinity, these two facts imply the convergence of  $F^*(t)$  to  $+\infty$ . As a consequence  $F^*(t)$  will become greater than  $n$  for  $t$  large enough which imply the first equality in (4.159). The second equality comes from the equivalence when  $t$  goes to infinity between  $(\tau(X, n) + t) \log(\log(\tau(X, n) + t))$  and  $t \log(\log(t))$ .

The limsup in the right hand side of (4.159) belongs to the  $\sigma$ -field  $\sigma(X^n(t), t \geq 0)$ , it is thus independent from  $\sigma(X(s), 0 \leq s \leq \tau(X, n))$ . Since this is true for any  $n \in \mathbb{N}$  we get, according to Kolmogorov 0 – 1 law, that the limsup is constant  $P^v$ -almost surely. In other words, the limsup is only a deterministic function of the environment  $v$ , let us denote it by  $L(v)$ .

Let us fix  $v \in \mathcal{V}$  and  $n \in \mathbb{N}$ . Note that  $P^v(\inf X^n \geq n - 1) > 0$  so the limsup still equals  $L(v)$  on  $\{\inf X^n \geq n - 1\}$ . As a consequence,  $L(v)$  is only a function of  $(v(x + n - 1) - v(n - 1), x \geq 0)$ . If we consider the space  $\mathcal{V}$  equipped with probability  $P$ , this implies that  $L(V)$  is independent from the  $\sigma$ -field  $\sigma(V(x), x \leq n - 1)$ . Since this is true for all  $n \in \mathbb{N}$  we deduce from Kolmogorov

0 – 1 law that  $L(V)$  is constant  $P$ -almost surely. This proves that the  $\limsup$  is in fact constant  $\mathbb{P}$ -almost surely. In other words,

$$\exists \lambda \in [0, +\infty] \text{ such that } \mathbb{P} - a.s. \limsup_{t \rightarrow +\infty} \frac{\mathcal{L}_X^*(t)}{t \log(\log(t))} = \lambda.$$

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